An Introduction to Categories and Sheaves

Pierre Schapira
Sorbonne Université
Institut de Mathématiques de Jussieu-Paris Rive Gauche, F-75005 Paris France
http://webusers.imj-prg.fr/~pierre.schapira
e-mail: pierre.schapira@imj-prg.fr

15/01/2023

Preface

Since more that half a century, the set theoretical point of view in mathematics has been supplanted by the categorical perspective. Category theory was introduced by Samuel Eilenberg and Saunders McLane at more or less at the same time as sheaf theory was by Jean Leray. Both theories, categories and sheaves, were incredibly developed by Alexander Grothendieck who made them the natural language for algebraic geometry, two cornerstones being first his famous Tohoku paper, second the introduction of the so-called “6 operations”. These new techniques are now basic in many fields, not only algebraic geometry, but also algebraic topology, analytic geometry, algebraic analysis and D-module theory, singularity theory, representation theory, etc. and, more recently, computational geometry.

The underlying idea of category theory is that mathematical objects only take their full force in relation with other objects of the same type. As we shall see, category theory is a very nice and natural language, not difficult to assimilate for any one having a bit of experience in mathematics, someone familiar with linear algebra and general topology. It opens new horizons in mathematics, a new way, a “functorial way”, of doing mathematics. Category theory reveals fundamental concepts and notions which across all mathematics, such as adjunction formulas, limits and colimits, or the difference between equalities and isomorphisms. And many theorems of today’s mathematics are simply expressed as equivalences of categories. The famous homological mirror symmetry, as formulated by Maxim Kontsevich, is a good illustration of this trend.

However, a difficulty soon appears: one should be careful with the size of the objects one manipulates and one is led to work in a given universe, changing of universe when necessary.

There is a class of categories which plays a central role: these are the additive categories and among them the abelian categories, in which one can perform homological algebra. Homological algebra is essentially linear algebra, no more over a field but over a ring and by extension, in abelian categories. When replacing a field with a ring, a submodule has in general no supplementary and the classical functors of tensor product and internal hom are no more exact and one has to consider their derived functors. Derived functors are of fundamental importance and many phenomena only appear in their light. Two classical examples are local cohomology of sheaves and duality. The calculation of the derived functor of a composition leads to technical difficulties, known as “spectral sequences”. Fortunately, the use of derived categories makes things much more elementary as we shall see in this book which never uses spectral sequences.

Sheaf theory is the mathematical tool to treat the familiar local/global dichotomy on topological spaces. More precisely, sheaf theory allows one to obtain obstructions to pass from local to global and, at the same time, to construct new objects which exist only globally. This theory is at full strength when working with abelian sheaves in the framework of derived categories. A good part of the theory (more precisely, four operations among six) may be naturally developed on sites, that is, categories endowed with Grothendieck topologies.

This text may be considered as an elementary introduction to the book [KS06] and in fact a few (as few as possible) difficult proofs, such as the Brown representability theorem, are omitted here, with the reader referred to that book. On the other hand, we study in some details sheaves on topological spaces and particularly on locally compact spaces, including duality, topics which are not treated in the book mentioned above.

A forthcoming manuscript. Initiated by Mikio Sato in the 70s, new perspectives appeared in analysis, known as microlocal analysis. The idea is to treat various phenomena on a real manifold M as the projection on M of phenomena living in the cotangent bundle T^*M. When applied to sheaf theory, one gets the "microlocal sheaf theory" introduced by Masaki Kashiwara and the author in 1982 and developed in [KS90]. This theory is, in some sense, the analogue on real manifolds of D-module theory and its microlocalization (see [SKK73, Kas03]) on complex manifolds. When combining both, microlocal sheaf theory and D-modules theory, most of classical results of analytic partial differential equations become easy exercises. Moreover, if microlocal sheaf theory uses basic results of symplectic geometry, conversely it can be an efficient tool to treat problems of symplectic topology (see [Tam08, NZ09, Gui19]).

We plan to come back to these subjects in a forthcoming manuscript, "Microlocal Algebraic Analysis".

Contents

Preface **3**

Introduction **9**

1 The language of categories **13**
- 1.1 Sets and maps . . . 13
- 1.2 Modules and linear maps . . . 15
- 1.3 Categories and functors . . . 19
- 1.4 The Yoneda Lemma . . . 25
- 1.5 Representable functors, adjoint functors . . . 27
- Exercises . . . 28

2 Limits **29**
- 2.1 Products and coproducts . . . 29
- 2.2 Kernels and cokernels . . . 31
- 2.3 Limits . . . 32
- 2.4 Fiber products and coproducts . . . 37
- 2.5 Properties of limits . . . 38
- 2.6 Filtered colimits . . . 40
- Exercises . . . 45

3 Localization **49**
- 3.1 Localization of categories . . . 49
- 3.2 Localization of subcategories . . . 54
- 3.3 Localization of functors . . . 55
- Exercises . . . 56

4 Additive categories **57**
- 4.1 Additive categories . . . 57
- 4.2 Complexes in additive categories . . . 58
- 4.3 Double complexes . . . 60
- 4.4 The homotopy category . . . 63
- 4.5 Simplicial constructions . . . 65
- Exercises . . . 67

5 Abelian categories **69**
- 5.1 Abelian categories . . . 69
- 5.2 Exact functors . . . 73
- 5.3 Injective and projective objects . . . 76

5.4 Generators and Grothendieck categories . . . 78
5.5 Complexes in abelian categories . . . 79
5.6 Double complexes in abelian categories . . . 84
5.7 Koszul complexes . . . 86
Exercises . . . 92

6 Triangulated categories 95
6.1 Triangulated categories . . . 95
6.2 Triangulated and cohomological functors . . . 97
6.3 Applications to the homotopy category . . . 99
6.4 Localization of triangulated categories . . . 99
Exercises . . . 103

7 Derived categories 105
7.1 Derived categories . . . 105
7.2 Resolutions . . . 108
7.3 Derived functors . . . 111
7.4 Bifunctors . . . 114
7.5 The Brown representability theorem . . . 119
Exercises . . . 122

8 Sheaves on sites 125
8.1 Abelian sheaves on topological spaces: a short introduction . . . 126
8.2 Presites and presheaves . . . 128
8.3 Operations on presheaves . . . 130
8.4 Grothendieck topologies . . . 135
8.5 Sheaves . . . 138
8.6 Sheaf associated with a presheaf . . . 140
8.7 Operations on sheaves . . . 144
8.8 Locally constant sheaves and glueing of sheaves . . . 149
Exercises . . . 152

9 Derived categories of abelian sheaves 155
9.1 Abelian sheaves . . . 155
9.2 Flat sheaves and injective sheaves . . . 158
9.3 The derived category of sheaves . . . 160
9.4 Modules over sheaves of rings . . . 162
9.5 Ringed sites . . . 164
Exercises . . . 165

10 Sheaves on topological spaces 167
10.1 Restriction of sheaves . . . 167
10.2 Sheaves associated with a locally closed subset . . . 169
10.3 Čech complexes for open coverings . . . 171
10.4 Čech complexes for closed coverings . . . 174
10.5 Flabby sheaves . . . 175
10.6 Sheaves on the interval $[0,1]$ . . . 177
10.7 Invariance by homotopy . . . 178
10.8 Action of groups . . . 181

10.9 Cohomology of some classical manifolds . . . 184
Exercises . . . 188

11 Duality on locally compact spaces 191
11.1 Proper direct images . . . 191
11.2 c-soft sheaves . . . 197
11.3 Derived proper direct images . . . 199
11.4 The functor $f^!$ . . . 202
11.5 Orientation and duality on $\mathscr{C}^0$-manifolds . . . 206
11.6 Cohomology of real and complex manifolds . . . 209
Exercises . . . 212

12 Constructible sheaves and the subanalytic topology 215
12.1 Subanalytic subsets . . . 215
12.2 Subanalytic geometry . . . 215
12.3 Constructible sheaves . . . 215
12.4 Subanalytic topology . . . 215

Bibliography 217

Introduction

These Notes contain roughly two parts: Chapters 1 to 7 treat homological algebra and Chapters 8 to 12 treat sheaf theory.

In a first part, we introduce the reader to the language of categories and we present the main notions of homological algebra, including triangulated and derived categories.

After having introduced the basic concepts and results of category theory, in particular the Yoneda lemma and the notion of representable functors, we define limits and colimits (also called projective and inductive limits). We start with the particular cases of kernels and products (and the dual notions of cokernels and co-products) and study filtered colimits. We also introduce, as a preparation for derived categories, the notion of localization of categories. Then we treat additive categories, complexes, shifted complexes, mapping cones and the homotopy category. In the course of the exposition, we introduce the truncation functors, an essential tool in practice and a substitute to the famous "spectral sequences" which shall not appear here. We also have a glance to simplicial constructions. Then we define abelian categories and study complexes in this framework. We admit without proof the Grothendieck theorem which asserts that abelian categories satisfying suitable properties admit enough injectives. We treat in some details Koszul complexes, giving many examples.

We introduce the basic notions on triangulated categories, study their localization and state, without proof, the Brown representability theorem. Finally we define and study derived categories and derived functors and bifunctors.

As it is well-known, it not possible to develop category theory without some caution about the size of the objects one considers. An easy and classical illustration of this fact is given in Remark 2.6.16. We shall not introduce cardinals, preferring to work with universes and referring to [KS06] for details. We shall mention when necessary (perhaps not always!) that a category is "small" or "big" with respect to a given universe $\mathscr{U}$, passing to a bigger universe if necessary. Notice that Grothendieck's theorem about the existence of injectives is a typical example (and historically, the first one) where universes play an essential role.

In a second part, we shall expose sheaf theory in the framework of derived categories. First, we study sheaves on sites, a site being a category endowed with a Grothendieck topology. We construct the operations of direct and inverse images and also introduce the important notion of locally constant sheaves and build them by glueing.

Abelian sheaves are sheaves with values in the category of abelian groups or more generally in the category of **k**-modules for a commutative unital ring **k**. We study the derived operations (tensor product and internal hom, direct and inverse images) on such sheaves and also briefly explain how to treat similarly sheaves of modules over a sheaf of rings. This theory is presented in the framework of sites but we restrict ourselves to the cases of categories admitting products and fiber products, which makes the theory much easier and very similar to that of sheaves on topological spaces.

Next we treat abelian sheaves on topological spaces. We use Čech cohomology and invariance by homotopy to calculate the cohomology of many classical manifolds. We have a glance on the action of (finite) groups, a way to calculate the cohomology of real projective spaces.

The theory becomes more fruitful on locally compact topological spaces. In this case, one defines the proper direct image functor and, using the Brown representability theorem, its right adjoint which only exists in the derived setting. This is a vaste generalization of Poincaré duality, known as Poincaré-Verdier duality. In the complex case, we explicitly construct the Leray-Grothendieck residues morphism. Here again, the natural formulation only makes sense in the derived setting.

Finally, we have a glance to $\mathbb{R}$-constructible sheaves on real analytic manifolds after a brief survey of subanalytic geometry.

Some historical comments and references.[1] As already mentioned, category theory was introduced by Samuel Eilenberg and Saunders McLane [EML45]. At the prehistory of homological algebra is the book [CE56] by Henri Cartan and Samuel Eilenberg including the Appendix by David A. Buchsbaum in which abelian categories are introduced for the first time, before being considerably developed and systematically studied by Gothendieck in [Gro57, SGA4]. The natural framework of homological algebra is that of derived categories, whose idea is, once more, due to Grothendieck and which was written done by his student, Jean-Louis Verdier who realized the importance of triangulated categories, a notion already used at that time in algebraic topology. Derived categories are constructed by "localizing" the homotopy category and the basic reference for localization is the book [GZ67] by Gabriel and Zisman. Category theory would not exist without the axiom of universes (or anything equivalent) and this is Grothendieck who introduced this axiom (see [SGA4]). We refer to [Krö07] for interesting thoughts on this topic.

Sheaf theory was created by Jean Leray when he was a war prisoner in the forties. Leray's ideas were not easy to follow, but were clarified by Henri Cartan and Jean-Pierre Serre in the fifties who made sheaf theory an essential tool for analytic and algebraic geometry at this time. We refer to the historical notes by Christian Houzel in [KS90]. After that, sheaf theory has been treated in the language of derived categories, allowing to construct (on suitable spaces) the extraordinary inverse image, a vaste generalization of Poincaré duality. Grothendieck constructed first this functor in the framework of étale topology and Verdier did it for locally compact topological spaces. Grothendieck also introduced what is now called "Grothendieck topologies" by remarking that there is no need of a topological space to develop sheaf theory and indeed, the usual notion of a topological space is not appropriate for algebraic geometry (there being an insufficiency of open subsets). The objects of any category may perfectly play the role of the open sets and it remains to define abstractly what are the coverings.

This text is largely inspired by [KS06], a book itself inspired by [SGA4]. Other references for category theory are [Mac98, Bor94] for the general theory, [MM92] for links with logic, [GM96, Wei94] and [KS90, Ch. 1] for homological algebra, including derived categories, as well as [Nee01, Yek20] for a more exhaustive study of triangulated categories and derived categories, the last reference developing the DG (differential graded) point of view.

---

[1] These few lines are not written by an historian of mathematics and should be read with caution.

Classical sheaf theory was first exposed in the book of Roger Godement [God58] then in [Bre67]. For an approach in the language of derived categories, see [Ive86, GM96, KS90, Dim04]. Sheaves on Grothendieck topologies are exposed in [SGA4] and [KS06]. A short presentation in case of the étale topology is given in [Tam94]. For constructible sheaves, see [KS90, ChVIII] (see also [Schu03]).

Prerequisites. The reader is supposed to have basic knowledges in algebra, general topology and real or complex analysis. In other words, to be familiar with the notions of modules over a ring, topological spaces and real or complex manifolds.

Conventions. In this book, all rings are unital and associative but not necessarily commutative. The operations, the zero element, and the unit are denoted by $+, \cdot, 0, 1$, respectively. However, we shall often write for short ab instead of $a \cdot b$. All along these Notes, $\mathbf{k}$ will denote a *commutative* ring of finite global dimension (see [Wei94, 4.1.2] or [KS90, Exe. I.28]). (Sometimes, $\mathbf{k}$ will be a field.) We denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. We denote by $\mathbb{N}$ the set of non-negative integers, $\mathbb{N} = \{0, 1, \dots\}$, by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the fields of rational numbers, real numbers and complex numbers, respectively. .

A comment: ∞-categories. These Notes are written in the “classical” language of category theory, that is, 1- or 2-categories. However, some specialists of algebraic geometry or algebraic topology are now using the language of ∞-categories, the homotopical approach replacing the homological one. The difficulty of this last theory is for the moment of another order of magnitude than that of the classical theory, this last one being perfectly suited for the applications we have in mind. References are made to [Cis19, Lan21, Lur09, Lur10, RV22, Toë14].

Chapter 1

The language of categories

Caution. All along this book, we shall be rather sketchy with the notion of universes, mentioning when necessary (perhaps not always!) that a category is "small" or "big" with respect to a universe $\mathscr{U}$. Indeed, it not possible to develop category theory without some caution about the size of the objects we consider. An easy and classical illustration of this fact is given in Remark 2.6.16.

In this chapter we start with some reminders on the categories **Set** of sets and $\mathrm{Mod}(A)$ of modules over a (not necessarily commutative) ring A. Then we expose the basic language of categories and functors. A key point is the Yoneda lemma, which asserts that a category $\mathscr{C}$ may be embedded in the category $\mathscr{C}^\wedge$ of contravariant functors on $\mathscr{C}$ with values in the category **Set**. This naturally leads to the concept of representable functor and adjoint functors.. Many examples are treated, in particular in the categories **Set** and $\mathrm{Mod}(A)$.

1.1 Sets and maps

The aim of this section is to fix some notations and to recall some elementary constructions on sets.

If $f\colon X \to Y$ is a map from a set X to a set Y, we shall often say that f is a morphism (of sets) from X to Y. We shall denote by $\mathrm{Hom}_{\mathbf{Set}}(X,Y)$, or simply $\mathrm{Hom}\,(X,Y)$ or also Y^X, the set of all maps from X to Y. If $g\colon Y \to Z$ is another map, we can define the composition $g \circ f\colon X \to Z$. Hence, we get two maps:

$$\begin{aligned} g\circ\colon \mathrm{Hom}\,(X,Y) &\to \mathrm{Hom}\,(X,Z), \\ \circ f\colon \mathrm{Hom}\,(Y,Z) &\to \mathrm{Hom}\,(X,Z). \end{aligned}$$

If f is bijective we shall say that f is an isomorphism and write $f\colon X \xrightarrow{\sim} Y$. This is equivalent to saying that there exists $g\colon Y \to X$ such that $g \circ f$ is the identity of X and $f \circ g$ is the identity of Y. If there exists an isomorphism $f\colon X \xrightarrow{\sim} Y$, we say that X and Y are isomorphic and write $X \simeq Y$.

Notice that if $X = \{x\}$ and $Y = \{y\}$ are two sets with one element each, then there exists a unique isomorphism $X \xrightarrow{\sim} Y$. Of course, if X and Y are finite sets with the same cardinal $\pi > 1$, X and Y are still isomorphic, but the isomorphism is no more unique.

In the sequel we shall denote by $\varnothing$ the empty set and by $\{\mathrm{pt}\}$ a set with one element. Note that for any set X, there is a unique map $\varnothing \to X$ and a unique map $X \to \{\mathrm{pt}\}$.

Let $\{X_i\}_{i\in I}$ be a family of sets indexed by a set I. Their union is denoted by $\bigcup_i X_i$. The product of the X_i's, denoted $\prod_{i\in I} X_i$, or simply $\prod_i X_i$, is defined as

$$\prod_{i\in I} X_i = \{f \in \operatorname{Hom}(I, \bigcup_i X_i); f(i) \in X_i \text{ for all } i \in I\}. \tag{1.1.1}$$

Hence, if $X_i = X$ for all $i \in I$, we get

$$\prod_{i\in I} X_i = \operatorname{Hom}(I, X) = X^I.$$

If I is the ordered set $\{1,2\}$, one sets

$$X_1 \times X_2 = \{(x_1, x_2); x_i \in X_i, i = 1, 2\}, \tag{1.1.2}$$

and there are natural isomorphisms

$$X_1 \times X_2 \simeq \prod_{i\in I} X_i \simeq X_2 \times X_1.$$

This notation and these isomorphisms extend to the case of a finite ordered set I.

If $\{X_i\}_{i\in I}$ is a family of sets indexed by a set I as above, one also considers their disjoint union, also called their coproduct. The coproduct of the X_i's is denoted $\bigsqcup_{i\in I} X_i$ or simply $\bigsqcup_i X_i$. If $X_i = X$ for all $i \in I$, one uses the notation $X^{\sqcup I}$. If $I = \{1,2\}$, one often writes $X_1 \sqcup X_2$ instead of $\bigsqcup_{i\in\{1,2\}} X_i$.

For three sets I, X, Y, there is a natural isomorphism

$$\operatorname{Hom}(I, \operatorname{Hom}(X, Y)) = \operatorname{Hom}(X, Y)^I \simeq \operatorname{Hom}(I \times X, Y). \tag{1.1.3}$$

For a set Y, there is a natural isomorphism

$$\operatorname{Hom}(Y, \prod_i X_i) \simeq \prod_i \operatorname{Hom}(Y, X_i). \tag{1.1.4}$$

Note that

$$X \times I \simeq X^{\sqcup I}. \tag{1.1.5}$$

For a set Y, there is a natural isomorphism

$$\operatorname{Hom}(\bigsqcup_i X_i, Y) \simeq \prod_i \operatorname{Hom}(X_i, Y). \tag{1.1.6}$$

In particular,

$$\operatorname{Hom}(X^{\sqcup I}, Y) \simeq \operatorname{Hom}(X, Y^I) \simeq \operatorname{Hom}(X, Y)^I.$$

Consider two sets X and Y and two maps f, g from X to Y. We write for short $f, g\colon X \rightrightarrows Y$. The kernel (or equalizer) of (f, g), denoted $\ker(f, g)$, is defined as

$$\ker(f, g) = \{x \in X; f(x) = g(x)\}. \tag{1.1.7}$$

Note that for a set Z, one has

$$\operatorname{Hom}(Z, \ker(f, g)) \simeq \ker(\operatorname{Hom}(Z, X) \rightrightarrows \operatorname{Hom}(Z, Y)). \tag{1.1.8}$$

Let us recall a few elementary definitions.

- A relation $\mathscr{R}$ on a set X is a subset of $X \times X$. One writes $x\mathscr{R}y$ if $(x, y) \in \mathscr{R}$.
- The opposite relation $\mathscr{R}^{\mathrm{op}}$ is defined by $x\mathscr{R}^{\mathrm{op}}y$ if and only if $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is reflexive if it contains the diagonal, that is, $x\mathscr{R}x$ for all $x \in X$.
- A relation $\mathscr{R}$ is symmetric if $x\mathscr{R}y$, then $y\mathscr{R}x$.
- A relation $\mathscr{R}$ is anti-symmetric if $x\mathscr{R}y$ and $y\mathscr{R}x$, then $x = y$.
- A relation $\mathscr{R}$ is transitive if $x\mathscr{R}y$ and $y\mathscr{R}z$, then $x\mathscr{R}z$.
- A relation $\mathscr{R}$ is an equivalence relation if it is reflexive, symmetric and transitive.
- A relation $\mathscr{R}$ is a pre-order if it is reflexive and transitive. A pre-order is often denoted $\leq$. If the pre-order is anti-symmetric, then one says that $\mathscr{R}$ is an order on X. A set endowed with a pre-order is called a partially ordered set, or, for short, a poset.
- Let $(I, \leq)$ be a poset. One says that $(I, \leq)$ is filtered (one also says "directed" or "filtrant") if I is non empty and for any $i, j \in I$ there exists k with $i \leq k$ and $j \leq k$.
- Assume $(I, \leq)$ is a filtered poset and let $J \subset I$ be a subset. One says that J is cofinal to I if for any $i \in I$ there exists $j \in J$ with $i \leq j$.

If $\mathscr{R}$ is a relation on a set X, there is a smallest equivalence relation which contains $\mathscr{R}$. (Take the intersection of all subsets of $X \times X$ which contain $\mathscr{R}$ and which are equivalence relations.)

Let $\mathscr{R}$ be an equivalence relation on a set X. A subset S of X is saturated if $x \in S$ and $x\mathscr{R}y$ implies $y \in S$. For $x \in X$, the smallest saturated subset $\widehat{x}$ of X containing x is called the equivalence class of x. One then defines a new set $X/\mathscr{R}$ and a canonical map $f \colon X \to X/\mathscr{R}$ as follows: the elements of $X/\mathscr{R}$ are the sets $\widehat{x}$ and the map f associates the set $\widehat{x}$ to $x \in X$.

1.2 Modules and linear maps

All along this book, a ring A means a unital associative ring, but A is not necessarily commutative. We shall denote by $\mathbf{k}$ a commutative ring. Recall that a $\mathbf{k}$-algebra A is a ring endowed with a morphism of rings $\varphi \colon \mathbf{k} \to A$ such that the image of $\mathbf{k}$ is contained in the center of A (*i.e.*, $\varphi(x)a = a\varphi(x)$ for any $x \in \mathbf{k}$ and $a \in A$). Notice that a ring A is always a $\mathbb{Z}$-algebra. If A is commutative, then A is an A-algebra.

Since we do not assume A commutative, we have to distinguish between left and right structures. Unless otherwise specified, a module M over A means a left A-module.

Recall that an A-module M is an additive group (whose operations and zero element are denoted $+, 0$) endowed with an external law $A \times M \to M$ (denoted

$(a, m) \mapsto a \cdot m$ or simply $(a, m) \mapsto am$) satisfying:

$$\begin{cases} (ab)m = a(bm) \\ (a+b)m = am + bm \\ a(m+m') = am + am' \\ 1 \cdot m = m \end{cases}$$

where $a, b \in A$ and $m, m' \in M$.

Note that, when A is a $\mathbf{k}$-algebra, M inherits a structure of a $\mathbf{k}$-module via φ. In the sequel, if there is no risk of confusion, we shall not write φ.

We denote by A^{op} the ring A with the opposite structure. Hence the product ab in A^{op} is the product ba in A and an A^{op}-module is a right A-module.

Note that if the ring A is a field (here, a field is always commutative), then an A-module is nothing but a vector space.

Examples 1.2.1. (i) The first example of a ring is $\mathbb{Z}$, the ring of integers. Since a field is a ring, $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are rings. If A is a commutative ring, then $A[x_1, \dots, x_n]$, the ring of polynomials in n variables with coefficients in A, is also a commutative ring. It is a sub-ring of $A[[x_1, \dots, x_n]]$, the ring of formal powers series with coefficients in A.

(ii) Let $\mathbf{k}$ be a field. For $n > 1$, the ring $M_n(\mathbf{k})$ of square matrices of rank n with entries in $\mathbf{k}$ is non-commutative.

(iii) Let $\mathbf{k}$ be a field. The *Weyl algebra* in n variables, denoted $W_n(\mathbf{k})$, is the non commutative ring of polynomials in the variables x_i, ∂_j $(1 \leq i, j \leq n)$ with coefficients in $\mathbf{k}$ and relations :

$$[x_i, x_j] = 0, \ [\partial_i, \partial_j] = 0, \ [\partial_j, x_i] = \delta_j^i$$

where $[p, q] = pq - qp$ and δ_j^i is the Kronecker symbol.

The Weyl algebra $W_n(\mathbf{k})$ may be regarded as the ring of differential operators with coefficients in $\mathbf{k}[x_1, \dots, x_n]$, and $\mathbf{k}[x_1, \dots, x_n]$ becomes a left $W_n(\mathbf{k})$-module: x_i acts by multiplication and ∂_i is the derivation with respect to x_i.

A morphism $f \colon M \to N$ of A-modules is an A-linear map, *i.e.,* f satisfies:

$$\begin{cases} f(m + m') = f(m) + f(m') & m, m' \in M, \\ f(am) = af(m) & m \in M, a \in A. \end{cases}$$

A morphism f is an isomorphism if there exists a morphism $g : N \to M$ with $f \circ g = \mathrm{id}_N, g \circ f = \mathrm{id}_M$.

If f is bijective, it is easily checked that the inverse map $f^{-1} : N \to M$ is itself A-linear. Hence f is an isomorphism if and only if f is A-linear and bijective.

A submodule N of M is a nonempty subset N of M such that if $n, n' \in N$, then $n + n' \in N$ and if $n \in N, a \in A$, then $an \in N$. A submodule of the A-module A is called an ideal of A. Note that if A is a field, it has no non-trivial ideal, *i.e.,* its only ideals are $\{0\}$ and A. If $A = \mathbb{C}[x]$, then $I = \{P \in \mathbb{C}[x]; P(0) = 0\}$ is a non trivial ideal.

If N is a submodule of M, it defines an equivalence relation: $m \mathscr{R} m'$ if and only if $m - m' \in N$. One easily checks that the quotient set $M/\mathscr{R}$ is naturally endowed with a structure of a left A-module. This module is called the quotient module and is denoted M/N.

Let $f\colon M \to N$ be a morphism of A-modules. One sets:

$$\begin{aligned} \ker f &= \{m \in M;\quad f(m) = 0\}, \\ \operatorname{Im} f &= \{n \in N;\quad \text{there exists } m \in M,\quad f(m) = n\}. \end{aligned}$$

These are submodules of M and N respectively, called the kernel and the image of f, respectively. One also introduces the cokernel and the coimage of f:

$$\operatorname{Coker} f = N/\operatorname{Im} f, \quad \operatorname{Coim} f = M/\ker f.$$

Note that the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Example 1.2.2. Let $W_n(\mathbf{k})$ denote as above the Weyl algebra. Consider the left $W_n(\mathbf{k})$-linear map $W_n(\mathbf{k}) \to \mathbf{k}[x_1, \dots, x_n]$, $W_n(\mathbf{k}) \ni P \mapsto P(1) \in \mathbf{k}[x_1, \dots, x_n]$. This map is clearly surjective and its kernel is the left ideal generated by $(\partial_1, \cdots, \partial_n)$. Hence, one has the isomorphism of left $W_n(\mathbf{k})$-modules:

$$W_n(\mathbf{k})/\sum_j W_n(\mathbf{k})\partial_j \xrightarrow{\sim} \mathbf{k}[x_1, \dots, x_n]. \tag{1.2.1}$$

Products and direct sums

Let I be a set, and let $\{M_i\}_{i\in I}$ be a family of A-modules indexed by I. The set $\prod_i M_i$ is naturally endowed with a structure of a left A-module by setting

$$\begin{aligned} (m_i)_i + (m'_i)_i &= (m_i + m'_i)_i, \\ a \cdot (m_i)_i &= (a \cdot m_i)_i. \end{aligned}$$

The direct sum $\bigoplus_i M_i$ is the submodule of $\prod_i M_i$ whose elements are the $(m_i)_i$'s such that $m_i = 0$ for all but a finite number of $i \in I$. In particular, if the set I is finite, we have $\bigoplus_i M_i = \prod_i M_i$. If $M_i = M$ for all i, one writes $M^{\oplus I}$ or $M^{(I)}$ instead of $\bigoplus_i M$.

Linear maps

Let M and N be two A-modules. Recall that an A-linear map $f\colon M \to N$ is also called a morphism of A-modules. One denotes by $\operatorname{Hom}_A(M, N)$ the set of A-linear maps $f\colon M \to N$. When A is a $\mathbf{k}$-algebra, $\operatorname{Hom}_A(M, N)$ is a $\mathbf{k}$-module. In fact one defines the action of $\mathbf{k}$ on $\operatorname{Hom}_A(M, N)$ by setting: $(\lambda f)(m) = \lambda(f(m))$. Hence $(\lambda f)(am) = \lambda f(am) = \lambda a f(m) = a\lambda f(m) = a(\lambda f)(m)$, and $\lambda f \in \operatorname{Hom}_A(M, N)$.

There is a natural isomorphism $\operatorname{Hom}_A(A, M) \simeq M$: to $u \in \operatorname{Hom}_A(A, M)$ one associates $u(1)$ and to $m \in M$ one associates the linear map $A \to M, a \mapsto am$. More generally, if I is an ideal of A then $\operatorname{Hom}_A(A/I, M) \simeq \{m \in M; Im = 0\}$.

Note that if A is a $\mathbf{k}$-algebra and $L \in \operatorname{Mod}(\mathbf{k})$, $M \in \operatorname{Mod}(A)$, the $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}}(L, M)$ is naturally endowed with a structure of a left A-module. If N is a right A-module, then $\operatorname{Hom}_{\mathbf{k}}(N, L)$ is naturally endowed with a structure of a left A-module.

Tensor product

Consider a right A-module N, a left A-module M and a $\mathbf{k}$-module L. Let us say that a map $f\colon N \times M \to L$ is $(A, \mathbf{k})$-bilinear if f is additive with respect to each of its arguments and satisfies $f(na, m) = f(n, am)$ and $f(n\lambda, m) = \lambda(f(n, m))$ for all $(n, m) \in N \times M$ and $a \in A, \lambda \in \mathbf{k}$.

Let us identify a set I to a subset of $\mathbf{k}^{(I)}$ as follows: to $i \in I$, we associate $\{l_j\}_{j\in I} \in \mathbf{k}^{(I)}$ given by

$$l_j = \begin{cases} 1 \text{ if} & j = i, \\ 0 \text{ if} & j \neq i. \end{cases} \tag{1.2.2}$$

The tensor product $N \otimes_A M$ is the $\mathbf{k}$-module defined as the quotient of $\mathbf{k}^{(N\times M)}$ by the submodule generated by the following elements (where $n, n' \in N, m, m' \in M, a \in A, \lambda \in \mathbf{k}$ and $N \times M$ is identified to a subset of $\mathbf{k}^{(N\times M)}$):

$$\begin{cases} (n + n', m) - (n, m) - (n', m), \\ (n, m + m') - (n, m) - (n, m'), \\ (na, m) - (n, am), \\ \lambda(n, m) - (n\lambda, m). \end{cases}$$

The image of (n, m) in $N \otimes_A M$ is denoted $n \otimes m$. Hence an element of $N \otimes_A M$ may be written (not uniquely!) as a finite sum $\sum_j n_j \otimes m_j$, $n_j \in N$, $m_j \in M$ and:

$$\begin{cases} (n + n') \otimes m = n \otimes m + n' \otimes m, \\ n \otimes (m + m') = n \otimes m + n \otimes m', \\ na \otimes m = n \otimes am, \\ \lambda(n \otimes m) = n\lambda \otimes m = n \otimes \lambda m. \end{cases}$$

Denote by $\beta\colon N \times M \to N \otimes_A M$ the natural map which associates $n \otimes m$ to (n, m).

Proposition 1.2.3. *The map β is $(A, \mathbf{k})$-bilinear and for any $\mathbf{k}$-module L and any $(A, \mathbf{k})$-bilinear map $f\colon N \times M \to L$, the map f factorizes uniquely through a $\mathbf{k}$-linear map $\varphi\colon N \otimes_A M \to L$.*

The proof is left to the reader.

Proposition 1.2.3 is visualized by the diagram:

$$\begin{tikzcd} N \times M \arrow[r, "\beta"] \arrow[rd, "f"'] & N \otimes_A M \arrow[d, dotted, "\varphi"] \\ & L. \end{tikzcd}$$

Consider an A-linear map $f\colon M \to L$. It defines a linear map $\mathrm{id}_N \times f\colon N \times M \to N \times L$, hence a $(A, \mathbf{k})$-bilinear map $N \times M \to N \otimes_A L$, and finally a $\mathbf{k}$-linear map

$$\mathrm{id}_N \otimes f\colon N \otimes_A M \to N \otimes_A L.$$

One constructs similarly $g \otimes \mathrm{id}_M$ associated to $g\colon N \to L$.

There are natural isomorphisms $A \otimes_A M \simeq M$ and $N \otimes_A A \simeq N$.

Denote by $\mathrm{Bil}(\mathrm{N}\times\mathrm{M},\mathrm{L})$ the $\mathbf{k}$-module of $(A,\mathbf{k})$-bilinear maps from $N\times M$ to L. One has the isomorphisms

$$(1.2.3)\qquad \begin{aligned}\mathrm{Bil}(\mathrm{N}\times\mathrm{M},\mathrm{L}) &\simeq \mathrm{Hom}_{\mathbf{k}}(N\otimes_A M, L)\\ &\simeq \mathrm{Hom}_{A}(M,\mathrm{Hom}_{\mathbf{k}}(N,L))\\ &\simeq \mathrm{Hom}_{A}(N,\mathrm{Hom}_{\mathbf{k}}(M,L)).\end{aligned}$$

For $L\in\mathrm{Mod}(\mathbf{k})$ and $M\in\mathrm{Mod}(A)$, the $\mathbf{k}$-module $L\otimes_{\mathbf{k}}M$ is naturally endowed with a structure of a left A-module. For $M,N\in\mathrm{Mod}(A)$ and $L\in\mathrm{Mod}(\mathbf{k})$, we have the isomorphisms (whose verification is left to the reader):

$$(1.2.4)\qquad \begin{aligned}\mathrm{Hom}_{A}(L\otimes_{\mathbf{k}}N,M) &\simeq \mathrm{Hom}_{A}(N,\mathrm{Hom}_{\mathbf{k}}(L,M))\\ &\simeq \mathrm{Hom}_{\mathbf{k}}(L,\mathrm{Hom}_{A}(N,M)).\end{aligned}$$

If A is commutative, $N\otimes_A M$ is naturally an A-module and there is an isomorphism: $N\otimes_A M\simeq M\otimes_A N$ given by $n\otimes m\mapsto m\otimes n$. Moreover, the tensor product is associative, that is, if L,M,N are A-modules, there are natural isomorphisms $L\otimes_A(M\otimes_A N)\simeq(L\otimes_A M)\otimes_A N$. One simply writes $L\otimes_A M\otimes_A N$.

1.3 Categories and functors

Definition 1.3.1. A category $\mathscr{C}$ consists of:

(i) a set $\mathrm{Ob}(\mathscr{C})$ whose elements are called the objects of $\mathscr{C}$,

(ii) for each $X,Y\in\mathrm{Ob}(\mathscr{C})$, a set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ whose elements are called the morphisms from X to Y,

(iii) for any $X,Y,Z\in\mathrm{Ob}(\mathscr{C})$, a map, called the composition, $\mathrm{Hom}_{\mathscr{C}}(X,Y)\times\mathrm{Hom}_{\mathscr{C}}(Y,Z)\to\mathrm{Hom}_{\mathscr{C}}(X,Z)$, and denoted $(f,g)\mapsto g\circ f$,

these data satisfying:

(a) $\circ$ is associative,

(b) for each $X\in\mathrm{Ob}(\mathscr{C})$, there exists $\mathrm{id}_X\in\mathrm{Hom}\,(X,X)$ such that for all $f\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $g\in\mathrm{Hom}_{\mathscr{C}}(Y,X)$, $f\circ\mathrm{id}_X=f$, $\mathrm{id}_X\circ g=g$.

Note that $\mathrm{id}_X\in\mathrm{Hom}\,(X,X)$ is characterized by the condition in (b).

Universes

With such a definition of a category, there is no category of sets, since there is no set of "all" sets. The set-theoretical dangers encountered in category theory will be illustrated in Remark 2.6.16.

To overcome this difficulty, one has to be more precise when using the word "set". One way is to use the notion of *universe*. We do not give in this book the precise definition of a universe, only recalling that a universe $\mathscr{U}$ is a set (a very big one) stable by many operations. In particular, $\varnothing\in\mathscr{U}$, $\mathbb{N}\in\mathscr{U}$, $x\in\mathscr{U}$ and $y\in x$ implies $y\in\mathscr{U}$, $x\in\mathscr{U}$ and $y\subset x$ implies $y\in\mathscr{U}$, if $I\in\mathscr{U}$ and $u_i\in\mathscr{U}$ for all $i\in I$, then $\bigcup_{i\in I}u_i\in\mathscr{U}$ and $\prod_{i\in I}u_i\in\mathscr{U}$. See for example [KS06, Def. 1.1.1].

Definition 1.3.2. Let $\mathscr{U}$ be a universe.

(a) A set E is a $\mathscr{U}$-set if it belongs to $\mathscr{U}$.

(b) A set E is $\mathscr{U}$-small if it is isomorphic to a $\mathscr{U}$-set.

(c) A $\mathscr{U}$-category $\mathscr{C}$ is a category such that for any $X, Y \in \mathscr{C}$, the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is $\mathscr{U}$-small.

(d) A $\mathscr{U}$-category $\mathscr{C}$ is $\mathscr{U}$-small if moreover the set $\mathrm{Ob}(\mathscr{C})$ is $\mathscr{U}$-small.

The crucial point is Grothendieck's axiom which says that any set belongs to some universe.

By a "big" category, we mean a category in a bigger universe. Note that, by Grothendieck's axiom, any category is an $\mathscr{V}$-category for a suitable universe $\mathscr{V}$ and one even can choose $\mathscr{V}$ so that $\mathscr{C}$ is $\mathscr{V}$-small.

As far as it has no implication, we shall not always be precise on this matter and the reader may skip the words "small" and "big".

Notation 1.3.3. One often writes $X \in \mathscr{C}$ instead of $X \in \mathrm{Ob}(\mathscr{C})$ and $f\colon X \to Y$ (or else $f\colon Y \leftarrow X$) instead of $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. One calls X the source and Y the target of f.

- A morphism $f\colon X \to Y$ is an *isomorphism* if there exists $g\colon X \leftarrow Y$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. In such a case, one writes $f\colon X \xrightarrow{\sim} Y$ or simply $X \simeq Y$. Of course g is unique, and one also denotes it by f^{-1}.
- A morphism $f\colon X \to Y$ is a *monomorphism* (resp. an *epimorphism*) if for any morphisms g_1 and g_2, $f \circ g_1 = f \circ g_2$ (resp. $g_1 \circ f = g_2 \circ f$) implies $g_1 = g_2$. One sometimes writes $f\colon X \rightarrowtail Y$ or else $X \hookrightarrow Y$ (resp. $f\colon X \twoheadrightarrow Y$) to denote a monomorphism (resp. an epimorphism).
- Two morphisms f and g are parallel if they have the same sources and targets, visualized by $f, g\colon X \rightrightarrows Y$.
- A category is *discrete* if the only morphisms are the identity morphisms. Note that a set is naturally identified with a discrete category (and conversely).
- A category $\mathscr{C}$ is *finite* if the family of all morphisms in $\mathscr{C}$ (hence, in particular, the family of objects) is a finite set.
- A category $\mathscr{C}$ is a *groupoid* if all morphisms are isomorphisms.

One introduces the *opposite category* $\mathscr{C}^{\mathrm{op}}$:

$$\mathrm{Ob}(\mathscr{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathscr{C}), \quad \mathrm{Hom}_{\mathscr{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X),$$

the identity morphisms and the composition of morphisms being the obvious ones.

A category $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$, denoted $\mathscr{C}' \subset \mathscr{C}$, if:

(a) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$,

(b) $\mathrm{Hom}_{\mathscr{C}'}(X,Y) \subset \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for any $X, Y \in \mathscr{C}'$, the composition $\circ$ in $\mathscr{C}'$ is induced by the composition in $\mathscr{C}$ and the identity morphisms in $\mathscr{C}'$ are induced by those in $\mathscr{C}$.

- One says that $\mathscr{C}'$ is a *full* subcategory if for all $X, Y \in \mathscr{C}'$, $\mathrm{Hom}_{\mathscr{C}'}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$.
- One says that a full subcategory $\mathscr{C}'$ of $\mathscr{C}$ is *saturated* if $X \in \mathscr{C}$ belongs to $\mathscr{C}'$ as soon as it is isomorphic to an object of $\mathscr{C}'$.

Examples 1.3.4. (i) **Set** is the category of sets and maps (in a given universe $\mathscr{U}$). If necessary, one calls this category $\mathscr{U}$-**Set**. Then $\mathbf{Set}^f$ is the full subcategory consisting of finite sets.
(ii) **Rel** is defined by: $\mathrm{Ob}(\mathbf{Rel}) = \mathrm{Ob}(\mathbf{Set})$ and $\mathrm{Hom}_{\mathbf{Rel}}(X,Y) = \mathscr{P}(X \times Y)$, the set of subsets of $X \times Y$. The composition law is defined as follows. For $f\colon X \to Y$ and $g\colon Y \to Z$, $g \circ f$ is the set

$$\{(x,z) \in X \times Z;\ \text{there exists } y \in Y \text{ with } (x,y) \in f, (y,z) \in g\}.$$

Of course, $\mathrm{id}_X = \Delta \subset X \times X$, the diagonal of $X \times X$.
(iii) Let A be a ring. The category of left A-modules and A-linear maps is denoted $\mathrm{Mod}(A)$. In particular $\mathrm{Mod}(\mathbb{Z})$ is the category of abelian groups.

We shall use the notation $\mathrm{Hom}_A(\,\bullet\,,\,\bullet\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(A)}(\,\bullet\,,\,\bullet\,)$.

One denotes by $\mathrm{Mod}^{\mathrm{f}}(A)$ the full subcategory of $\mathrm{Mod}(A)$ consisting of finitely generated A-modules.
(iv) One associates to a pre-ordered set $(I, \leq)$ a category, still denoted by I for short, as follows. $\mathrm{Ob}(I) = I$, and the set of morphisms from i to j has a single element if $i \leq j$, and is empty otherwise. Note that I^{op} is the category associated with I endowed with the opposite pre-order.
(v) We denote by **Top** the category of topological spaces and continuous maps.
(vi) We shall often represent by the diagram $\bullet \to \bullet$ the category which consists of two objects, say $\{a, b\}$, and one morphism $a \to b$ other than id_a and id_b. We denote this category by **Arr**.
(vii) We represent by $\bullet \rightrightarrows \bullet$ the category with two objects, say $\{a, b\}$, and two parallel morphisms $a \rightrightarrows b$ other than id_a and id_b.
(viii) Let G be a group. We may attach to it the groupoid $\mathscr{G}$ with one object, say $\{a\}$ and morphisms $\mathrm{Hom}_{\mathscr{G}}(a,a) = G$.
(ix) Let X be a topological space locally arcwise connected. We attach to it a category $\widetilde{X}$ as follows: $\mathrm{Ob}(\widetilde{X}) = X$ and for $x, y \in X$, a morphism $f\colon x \to y$ is a path form x to y. (Precise definitions are left to the reader.)

Definition 1.3.5. (i) An object $P \in \mathscr{C}$ is called initial if $\mathrm{Hom}_{\mathscr{C}}(P,X) \simeq \{\mathrm{pt}\}$ for all $X \in \mathscr{C}$. One often denotes by $\varnothing_{\mathscr{C}}$ an initial object in $\mathscr{C}$.

(ii) One says that P is terminal if P is initial in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X,P) \simeq \{\mathrm{pt}\}$. One often denotes by $\mathrm{pt}_{\mathscr{C}}$ a terminal object in $\mathscr{C}$.

(iii) One says that P is a zero-object if it is both initial and terminal. In such a case, one often denotes it by 0. If $\mathscr{C}$ has a zero object, for any objects $X, Y \in \mathscr{C}$, the morphism obtained as the composition $X \to 0 \to Y$ is still denoted by $0\colon X \to Y$.

Note that initial (resp. terminal) objects are unique up to unique isomorphisms.

Examples 1.3.6. (i) In the category **Set**, $\varnothing$ is initial and $\{\mathrm{pt}\}$ is terminal.

(ii) The zero module 0 is a zero-object in $\mathrm{Mod}(A)$.

(iii) The category associated with the ordered set $(\mathbb{Z}, \leq)$ has neither initial nor terminal object.

Definition 1.3.7. Let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ consists of a map $F\colon \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{C}')$ and for all $X, Y \in \mathscr{C}$, of a map still denoted by F: $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ such that

$$F(\mathrm{id}_X) = \mathrm{id}_{F(X)}, \quad F(f \circ g) = F(f) \circ F(g).$$

A contravariant functor from $\mathscr{C}$ to $\mathscr{C}'$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to $\mathscr{C}'$. In other words, it satisfies $F(g \circ f) = F(f) \circ F(g)$. If one wishes to put the emphasis on the fact that a functor is not contravariant, one says it is covariant.

One denotes by $\mathrm{op} : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ the contravariant functor, associated with $\mathrm{id}_{\mathscr{C}^{\mathrm{op}}}$.

Example 1.3.8. Let $\mathscr{C}$ be a category and let $X \in \mathscr{C}$.

(i) $\mathrm{Hom}_{\mathscr{C}}(X, \bullet)$ is a functor from $\mathscr{C}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(X,Y)$ and to a morphism $f\colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(X,f)\colon \mathrm{Hom}_{\mathscr{C}}(X,Y) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(X,Z).$$

(ii) $\mathrm{Hom}_{\mathscr{C}}(\bullet, X)$ is a functor from $\mathscr{C}^{\mathrm{op}}$ to **Set**. To $Y \in \mathscr{C}$, it associates the set $\mathrm{Hom}_{\mathscr{C}}(Y,X)$ and to a morphism $f\colon Y \to Z$ in $\mathscr{C}$, it associates the map

$$\mathrm{Hom}_{\mathscr{C}}(f,X)\colon \mathrm{Hom}_{\mathscr{C}}(Z,X) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(Y,X).$$

Example 1.3.9. Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A)$. Similarly as in Example 1.3.8, we have the functors

$$\mathrm{Hom}_A(M, \bullet)\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),$$
$$\mathrm{Hom}_A(\bullet, M)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k})$$

Clearly, the functor $\mathrm{Hom}_A(M, \bullet)$ commutes with products in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(M, \prod_i N_i) \simeq \prod_i \mathrm{Hom}_A(M, N_i)$$

and the functor $\mathrm{Hom}_A(\bullet, N)$ commutes with direct sums in $\mathrm{Mod}(A)$, that is,

$$\mathrm{Hom}_A(\bigoplus_i M_i, N) \simeq \prod_i \mathrm{Hom}_A(M_i, N).$$

(ii) Let N be a right A-module. Then $N \otimes_A \bullet\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is a functor. Clearly, the functor $N \otimes_A \bullet$ commutes with direct sums, that is,

$$N \otimes_A (\bigoplus_i M_i) \simeq \bigoplus_i (N \otimes_A M_i),$$

and similarly with the functor $\bullet \otimes_A M$.

Definition 1.3.10. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) One says that F is faithful (resp. full, resp. fully faithful) if for $X, Y \in \mathscr{C}$ $\mathrm{Hom}_{\mathscr{C}}(X,Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X),F(Y))$ is injective (resp. surjective, resp. bijective).

(ii) One says that F is essentially surjective if for each $Y \in \mathscr{C}'$ there exists $X \in \mathscr{C}$ and an isomorphism $F(X) \simeq Y$.

(iii) One says that F is conservative if any morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $F(f)$ is an isomorphism.

Examples 1.3.11. (i) The forgetful functor *for*: $\mathrm{Mod}(A) \to \mathbf{Set}$ associates to an A-module M the set M, and to a linear map f the map f. The functor *for* is faithful and conservative but not fully faithful.
(ii) The forgetful functor *for*: $\mathbf{Top} \to \mathbf{Set}$ (defined similarly as in (i)) is faithful. It is neither fully faithful nor conservative.
(iii) Consider the functor *for*: $\mathbf{Set} \to \mathbf{Rel}$ which is the identity on the objects of these categories and which, to a morphism $f\colon X \to Y$ in $\mathbf{Set}$ associates its graph $\Gamma_f \subset X\times Y$. This forgetful functor is faithful but not fully faithful. It is conservative (this is left as an exercise).

One defines the product of two categories $\mathscr{C}$ and $\mathscr{C}'$ by :

$$\begin{aligned}\mathrm{Ob}(\mathscr{C}\times\mathscr{C}') &= \mathrm{Ob}(\mathscr{C})\times\mathrm{Ob}(\mathscr{C}')\\ \mathrm{Hom}_{\mathscr{C}\times\mathscr{C}'}((X,X'),(Y,Y')) &= \mathrm{Hom}_{\mathscr{C}}(X,Y)\times\mathrm{Hom}_{\mathscr{C}'}(X',Y').\end{aligned}$$

A bifunctor $F\colon \mathscr{C}\times\mathscr{C}' \to \mathscr{C}''$ is a functor on the product category. This means that for $X \in \mathscr{C}$ and $X' \in \mathscr{C}'$, $F(X,\bullet)\colon \mathscr{C}' \to \mathscr{C}''$ and $F(\bullet, X') : \mathscr{C} \to \mathscr{C}''$ are functors, and moreover for any morphisms $f : X \to Y$ in $\mathscr{C}$, $g : X' \to Y'$ in $\mathscr{C}'$, the diagram below commutes:

$$\begin{tikzcd}
F(X,X') \arrow[r, "F(X,g)"] \arrow[d, "F(f,X')"'] & F(X,Y') \arrow[d, "F(f,Y')"] \\
F(Y,X') \arrow[r, "F(Y,g)"] & F(Y,Y')
\end{tikzcd}$$

In fact, $(f,g) = (\mathrm{id}_Y, g)\circ(f,\mathrm{id}_{X'}) = (f,\mathrm{id}_{Y'})\circ(\mathrm{id}_X,g)$.

Examples 1.3.12. (i) $\mathrm{Hom}_{\mathscr{C}}(\bullet,\bullet) : \mathscr{C}^{\mathrm{op}}\times\mathscr{C} \to \mathbf{Set}$ is a bifunctor.
(ii) If A is a $\mathbf{k}$-algebra, we have met the bifunctors

$$\begin{aligned}\mathrm{Hom}_A(\bullet,\bullet) &: \mathrm{Mod}(A)^{\mathrm{op}}\times\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}),\\ \bullet\otimes_A\bullet &: \mathrm{Mod}(A^{\mathrm{op}})\times\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k}).\end{aligned}$$

Definition 1.3.13. Let F_1, F_2 be two functors from $\mathscr{C}$ to $\mathscr{C}'$. A morphism of functors $\theta\colon F_1 \to F_2$ is the data for all $X \in \mathscr{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f\colon X \to Y$, the diagram below commutes:

(1.3.1)
$$\begin{tikzcd}
F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(f)"'] & F_2(X) \arrow[d, "F_2(f)"] \\
F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y).
\end{tikzcd}$$

A morphism of functors is visualized by a diagram:

$$\begin{tikzcd} \mathscr{C} \arrow[r, bend left, "F_1"{name=U}] \arrow[r, bend right, "F_2"'{name=D}] & \mathscr{C}' \arrow[from=U, to=D, Rightarrow, "\theta"] \end{tikzcd}$$

Hence, by considering the family of functors from $\mathscr{C}$ to $\mathscr{C}'$ and the morphisms of such functors, we get a new category.

Notation 1.3.14. (i) We denote by $\mathrm{Fct}(\mathscr{C},\mathscr{C}')$ the category of functors from $\mathscr{C}$ to $\mathscr{C}'$. One may also use the shorter notation $(\mathscr{C}')^{\mathscr{C}}$.

Examples 1.3.15. Let $\mathbf{k}$ be a field and consider the functor

$$\begin{aligned} &{}^*\colon \mathrm{Mod}(\mathbf{k})^{\mathrm{op}} \to \mathrm{Mod}(\mathbf{k}),\\ &V \mapsto V^* = \mathrm{Hom}_{\mathbf{k}}(V,\mathbf{k}), \quad u\colon V \to W \mapsto u^*\colon W^* \to V^*. \end{aligned}$$

Then there is a morphism of functors $\mathrm{id} \to {}^* \circ {}^*$ in $\mathrm{Fct}(\mathrm{Mod}(\mathbf{k}),\mathrm{Mod}(\mathbf{k}))$. Indeed, for any $V \in \mathrm{Mod}(\mathbf{k})$, there is a natural morphism $V \to V^{**}$ and for $u\colon V \to W$ a linear map, the diagram below commutes:

$$\begin{tikzcd} V \arrow[r] \arrow[d, "u"'] & V^{**} \arrow[d, "u^{**}"] \\ W \arrow[r] & W^{**}. \end{tikzcd} \tag{1.3.2}$$

(ii) We shall encounter morphisms of functors when considering pairs of adjoint functors (see (1.5.2)).

In particular we have the notion of an isomorphism of categories. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an isomorphism of categories if there exists $G : \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F = \mathrm{id}_{\mathscr{C}}$ and $F \circ G = \mathrm{id}_{\mathscr{C}'}$. In particular, for all $X \in \mathscr{C}$, $G \circ F(X) = X$. In practice, such a situation rarely occurs and is not really interesting. There is a weaker notion that we introduce below.

Definition 1.3.16. A functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if there exists $G\colon \mathscr{C}' \to \mathscr{C}$ such that: $G \circ F$ is isomorphic to $\mathrm{id}_{\mathscr{C}}$ and $F \circ G$ is isomorphic to $\mathrm{id}_{\mathscr{C}'}$.

We shall not give the proof of the following important result below.

Theorem 1.3.17. *The functor $F\colon \mathscr{C} \to \mathscr{C}'$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

If two categories are equivalent, all results and concepts in one of them have their counterparts in the other one. This is why this notion of equivalence of categories plays an important role in Mathematics.

Examples 1.3.18. (i) Let $\mathbf{k}$ be a field and let $\mathscr{C}$ denote the category defined by $\mathrm{Ob}(\mathscr{C}) = \mathbb{N}$ and $\mathrm{Hom}_{\mathscr{C}}(n,m) = M_{m,n}(\mathbf{k})$, the space of matrices of type (m,n) with entries in a field $\mathbf{k}$ (the composition being the usual composition of matrices). Define the functor $F\colon \mathscr{C} \to \mathrm{Mod}^f(\mathbf{k})$ as follows. To $n \in \mathbb{N}$, $F(n)$ associates $\mathbf{k}^n \in \mathrm{Mod}^f(\mathbf{k})$ and to a matrix of type (m,n), F associates the induced linear map from $\mathbf{k}^n$ to $\mathbf{k}^m$.

Clearly F is fully faithful. Since any finite dimensional vector space admits a basis, it is isomorphic to $\mathbf{k}^n$ for some n, hence F is essentially surjective. In conclusion, F is an equivalence of categories.

(ii) let $\mathscr{C}$ and $\mathscr{C}'$ be two categories. There is an equivalence

$$\mathrm{Fct}(\mathscr{C},\mathscr{C}')^{\mathrm{op}} \simeq \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},\mathscr{C}'^{\mathrm{op}}). \tag{1.3.3}$$

(iii) Let I, J and $\mathscr{C}$ be categories. There are equivalences

$$\mathrm{Fct}(I\times J,\mathscr{C}) \simeq \mathrm{Fct}(J,\mathrm{Fct}(I,\mathscr{C})) \simeq \mathrm{Fct}(I,\mathrm{Fct}(J,\mathscr{C})). \tag{1.3.4}$$

1.4 The Yoneda Lemma

Definition 1.4.1. Let $\mathscr{C}$ be a category. One defines the big categories

$$\mathscr{C}^\wedge = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},\mathbf{Set}),\ \mathscr{C}^\vee = \mathrm{Fct}(\mathscr{C}^{\mathrm{op}},\mathbf{Set}^{\mathrm{op}}),$$

and the functors

$$\begin{aligned}\mathrm{h}_{\mathscr{C}} &: \mathscr{C}\to\mathscr{C}^\wedge, \quad X\mapsto \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,,X)\\ \mathrm{k}_{\mathscr{C}} &: \mathscr{C}\to\mathscr{C}^\vee, \quad X\mapsto \mathrm{Hom}_{\mathscr{C}}(X,\,\bullet\,).\end{aligned}$$

Since there is a natural equivalence of categories

$$\mathscr{C}^\vee \simeq \mathscr{C}^{\mathrm{op},\wedge,\mathrm{op}}, \tag{1.4.1}$$

we shall concentrate our study on $\mathscr{C}^\wedge$.

Theorem 1.4.2. (The Yoneda lemma.) *For $A\in\mathscr{C}^\wedge$ and $X\in\mathscr{C}$, there is an isomorphism $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X),A)\simeq A(X)$, functorial with respect to X and A.*

Proof. One constructs the morphism $\varphi\colon \mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X),A)\to A(X)$ by the chain of morphisms: $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X),A)\to\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}_{\mathscr{C}}(X,X),A(X))\to A(X)$, where the last map is associated with id_X.

To construct $\psi\colon A(X)\to\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X),A)$, it is enough to associate with $s\in A(X)$ and $Y\in\mathscr{C}$ a map from $\mathrm{Hom}_{\mathscr{C}}(Y,X)$ to $A(Y)$. It is defined by the chain of maps $\mathrm{Hom}_{\mathscr{C}}(Y,X)\to\mathrm{Hom}_{\mathbf{Set}}(A(X),A(Y))\to A(Y)$ where the last map is associated with $s\in A(X)$.

One checks that φ and ψ are inverse to each other. □

Corollary 1.4.3. *The functors $\mathrm{h}_{\mathscr{C}}$ and $\mathrm{k}_{\mathscr{C}}$ are fully faithful.*

Proof. For $X,Y\in\mathscr{C}$, one has $\mathrm{Hom}_{\mathscr{C}^\wedge}(\mathrm{h}_{\mathscr{C}}(X),\mathrm{h}_{\mathscr{C}}(Y))\simeq\mathrm{h}_{\mathscr{C}}(Y)(X)=\mathrm{Hom}_{\mathscr{C}}(X,Y)$. □

One calls $\mathrm{h}_{\mathscr{C}}$ and $\mathrm{k}_{\mathscr{C}}$ the Yoneda embeddings.

Hence, one may consider $\mathscr{C}$ as a full subcategory of $\mathscr{C}^\wedge$. In particular, for $X\in\mathscr{C}$, $\mathrm{h}_{\mathscr{C}}(X)$ determines X up to unique isomorphism, that is, an isomorphism $\mathrm{h}_{\mathscr{C}}(X)\simeq\mathrm{h}_{\mathscr{C}}(Y)$ determines a unique isomorphism $X\simeq Y$.

Corollary 1.4.4. *Let $\mathscr{C}$ be a category and let $f\colon X\to Y$ be a morphism in $\mathscr{C}$.*

(i) *Assume that for any* $Z \in \mathscr{C}$, *the map* $\mathrm{Hom}_{\mathscr{C}}(Z,X) \xrightarrow{f\circ} \mathrm{Hom}_{\mathscr{C}}(Z,Y)$ *is bijective. Then* f *is an isomorphism.*

(ii) *Assume that for any* $Z \in \mathscr{C}$, *the map* $\mathrm{Hom}_{\mathscr{C}}(Y,Z) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X,Z)$ *is bijective. Then* f *is an isomorphism.*

Proof. (i) By the hypothesis, $\mathrm{h}_{\mathscr{C}}(f) : \mathrm{h}_{\mathscr{C}}(X) \to \mathrm{h}_{\mathscr{C}}(Y)$ is an isomorphism in $\mathscr{C}^\wedge$. Since $\mathrm{h}_{\mathscr{C}}$ is fully faithful, this implies that f is an isomorphism (see Exercise 1.2 (ii)).
(ii) follows by replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$. ☐

Definition 1.4.5. Let $\mathscr{C}$ and $\mathscr{C}'$ be categories, $F\colon \mathscr{C} \to \mathscr{C}'$ a functor and let $Z \in \mathscr{C}'$.

(i) The category $\mathscr{C}_Z$ is defined as follows:

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}_Z) &= \{(X,u); X \in \mathscr{C}, u\colon F(X) \to Z\},\\ \mathrm{Hom}_{\mathscr{C}_Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_1 = u_2 \circ F(v)\}. \end{aligned}$$

(ii) The category $\mathscr{C}^Z$ is defined as follows:

$$\begin{aligned} \mathrm{Ob}(\mathscr{C}^Z) &= \{(X,u); X \in \mathscr{C}, u\colon Z \to F(X))\},\\ \mathrm{Hom}_{\mathscr{C}^Z}((X_1,u_1),(X_2,u_2)) &= \{v\colon X_1 \to X_2; u_2 = u_1 \circ F(v)\}. \end{aligned}$$

Note that the natural functors $(X,u) \mapsto X$ from $\mathscr{C}_Z$ and $\mathscr{C}^Z$ to $\mathscr{C}$ are faithful.

The morphisms in $\mathscr{C}_Z$ (resp. $\mathscr{C}^Z$) are visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{tikzcd} F(X_1) \arrow[r, "u_1"] \arrow[d, "F(v)"'] & Z \\ F(X_2) \arrow[ur, "u_2"'] & \end{tikzcd} \qquad \begin{tikzcd} Z \arrow[r, "u_1"] \arrow[dr, "u_2"'] & F(X_1) \arrow[d, "F(v)"] \\ & F(X_2) \end{tikzcd}$$

Definition 1.4.6. Let $\mathscr{C}$ be a category. The category $\mathrm{Mor}(\mathscr{C})$ of morphisms in $\mathscr{C}$ is defined as follows.

$$\begin{aligned} &\mathrm{Ob}(\mathrm{Mor}(\mathscr{C})) = \{(U,V,s); U,V \in \mathscr{C}_X, s \in \mathrm{Hom}_{\mathscr{C}}(U,V),\\ &\mathrm{Hom}_{\,\mathrm{Mor}(\mathscr{C})}((s : U \to V),(s' : U' \to V')\\ &\qquad\qquad = \{u : U \to U', v : V \to V'; v \circ s = s' \circ u\}. \end{aligned}$$

The category $\mathrm{Mor}_0(\mathscr{C})$ is defined as follows.

$$\begin{aligned} &\mathrm{Ob}(\mathrm{Mor}_0(\mathscr{C})) = \{(U,V,s); U,V \in \mathscr{C}_X, s \in \mathrm{Hom}_{\mathscr{C}}(U,V),\\ &\mathrm{Hom}_{\,\mathrm{Mor}_0(\mathscr{C})}((s : U \to V),(s' : U' \to V')\\ &\qquad\qquad = \{u : U \to U', w : V' \to V; s = w \circ s' \circ u\}. \end{aligned}$$

A morphism $(s\colon U \to V) \to (s'\colon U' \to V')$ in $\mathrm{Mor}(\mathscr{C})$ (resp. $\mathrm{Mor}_0(\mathscr{C})$) is visualized by the commutative diagram on the left (resp. on the right) below:

$$\begin{tikzcd} U \arrow[r, "s"] \arrow[d, "u"'] & V \arrow[d, "v"] \\ U' \arrow[r, "s'"] & V', \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "s"] \arrow[d, "u"'] & V \\ U' \arrow[r, "s'"] & V'. \arrow[u, "w"'] \end{tikzcd}$$

1.5 Representable functors, adjoint functors

Representable functors

Definition 1.5.1. (i) One says that a functor F from $\mathscr{C}^{\mathrm{op}}$ to **Set** is representable if there exists $X \in \mathscr{C}$ such that $F(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(Y, X)$ functorially in $Y \in \mathscr{C}$. In other words, $F \simeq \mathrm{h}_{\mathscr{C}}(X)$ in $\mathscr{C}^{\wedge}$. Such an object X is called a representative of F.

(ii) Similarly, a functor $G\colon \mathscr{C} \to \mathbf{Set}$ is representable if there exists $X \in \mathscr{C}$ such that $G(Y) \simeq \mathrm{Hom}_{\mathscr{C}}(X, Y)$ functorially in $Y \in \mathscr{C}$.

It is important to notice that the isomorphisms above determine X up to unique isomorphism. More precisely, given two isomorphisms $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X)$ and $F \xrightarrow{\sim} \mathrm{h}_{\mathscr{C}}(X')$ there exists a unique isomorphism $\theta\colon X \xrightarrow{\sim} X'$ making the diagram below commutative:

$$\begin{tikzcd}
 & F \arrow[ld, "\sim"'] \arrow[rd, "\sim"] & \\
\mathrm{h}_{\mathscr{C}}(X) \arrow[rr, "\mathrm{h}_{\mathscr{C}}(\theta)", "\sim"'] & & \mathrm{h}_{\mathscr{C}}(X').
\end{tikzcd}$$

Representable functors provides a categorical language to deal with universal problems. Let us illustrate this by an example.

Example 1.5.2. Let A be a $\mathbf{k}$-algebra. Let N be a right A-module, M a left A-module and L a $\mathbf{k}$-module. Denote by $B(N \times M, L)$ the set of $(A, \mathbf{k})$-bilinear maps from $N \times M$ to L. Then the functor $F\colon L \mapsto B(N \times M, L)$ is representable by $N \otimes_A M$ by (1.2.3).

Adjoint functors

Definition 1.5.3. Let $F\colon \mathscr{C} \to \mathscr{C}'$ and $G\colon \mathscr{C}' \to \mathscr{C}$ be two functors. One says that (F, G) is a pair of adjoint functors or that F is a left adjoint to G, or that G is a right adjoint to F if there exists an isomorphism of bifunctors:

$$\mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), \,\bullet\,) \simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G(\,\bullet\,)). \tag{1.5.1}$$

If G is an adjoint to F, then G is unique up to isomorphism. In fact, $G(Y)$ is a representative of the functor $X \mapsto \mathrm{Hom}_{\mathscr{C}'}(F(X), Y)$.

The isomorphism (1.5.1) gives the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}'}(F \circ G(\,\bullet\,), \,\bullet\,) &\simeq \mathrm{Hom}_{\mathscr{C}}(G(\,\bullet\,), G(\,\bullet\,)),\\
\mathrm{Hom}_{\mathscr{C}'}(F(\,\bullet\,), F(\,\bullet\,)) &\simeq \mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, G \circ F(\,\bullet\,)).
\end{aligned}$$

In particular, we have morphisms $X \to G \circ F(X)$, functorial in $X \in \mathscr{C}$, and morphisms $F \circ G(Y) \to Y$, functorial in $Y \in \mathscr{C}'$. In other words, we have morphisms of functors

$$F \circ G \to \mathrm{id}_{\mathscr{C}'}, \qquad \mathrm{id}_{\mathscr{C}} \to G \circ F. \tag{1.5.2}$$

Examples 1.5.4. (i) Let $X \in \mathbf{Set}$. Using the bijection (1.1.3), we get that the functor $\mathrm{Hom}_{\mathbf{Set}}(X, \bullet) \colon \mathbf{Set} \to \mathbf{Set}$ is right adjoint to the functor $\bullet \times X$.

(ii) Let A be a $\mathbf{k}$-algebra and let $L \in \mathrm{Mod}(\mathbf{k})$. Using the first isomorphism in (1.2.4), we get that the functor $\mathrm{Hom}_{\mathbf{k}}(L, \bullet) \colon \mathrm{Mod}(A)$ to $\mathrm{Mod}(A)$ is right adjoint to the functor $\bullet \otimes_{\mathbf{k}} L$.

(iii) Let A be a $\mathbf{k}$-algebra. Using the isomorphisms in (1.2.4) with $N = A$, we get that the functor $\mathit{for} \colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ which, to an A-module associates the underlying $\mathbf{k}$-module, is right adjoint to the functor $A \otimes_{\mathbf{k}} \bullet \colon \mathrm{Mod}(\mathbf{k}) \to \mathrm{Mod}(A)$ (extension of scalars).

Exercises to Chapter 1

Exercise 1.1. Prove that the categories $\mathbf{Set}$ and $\mathbf{Set}^{\mathrm{op}}$ are not equivalent and similarly with the categories $\mathbf{Set}^f$ and $(\mathbf{Set}^f)^{\mathrm{op}}$.

(Hint: if $F \colon \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$ were such an equivalence, then $F(\varnothing) \simeq \{\mathrm{pt}\}$ and $F(\{\mathrm{pt}\}) \simeq \varnothing$. Now compare $\mathrm{Hom}_{\mathbf{Set}}(\{\mathrm{pt}\}, X)$ and $\mathrm{Hom}_{\mathbf{Set}^{\mathrm{op}}}(F(\{\mathrm{pt}\}), F(X))$ when X is a set with two elements.)

Exercise 1.2. (i) Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a faithful functor and let f be a morphism in $\mathscr{C}$. Prove that if $F(f)$ is a monomorphism (resp. an epimorphism), then f is a monomorphism (resp. an epimorphism).

(ii) Assume now that F is fully faithful. Prove that if $F(f)$ is an isomorphism, then f is an isomorphism. In other words, fully faithful functors are conservative.

Exercise 1.3. Is the natural functor $\mathbf{Set} \to \mathbf{Rel}$ full, faithful, fully faithful, conservative?

Exercise 1.4. Prove that the category $\mathscr{C}$ is equivalent to the opposite category $\mathscr{C}^{\mathrm{op}}$ in the following cases:

(i) $\mathscr{C}$ denotes the category of finite abelian groups,

(ii) $\mathscr{C}$ is the category $\mathbf{Rel}$ of relations.

Exercise 1.5. (i) Prove that in the category $\mathbf{Set}$, a morphism f is a monomorphism (resp. an epimorphism) if and only if it is injective (resp. surjective).

(ii) Prove that in the category of rings, the morphism $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism. (Hint: if $f \colon \mathbb{Q} \to A$ is a morphism of rings, then $f(p/q) = f(p) \times f(q)^{-1}$.)

(iii) In the category $\mathbf{Top}$, give an example of a morphism which is both a monomorphism and an epimorphism and which is not an isomorphism.

Exercise 1.6. Let $F \colon \mathscr{C} \to \mathbf{Set}$ be a functor and let $u \colon X \to Y$ be a morphisms in $\mathscr{C}$. Assume that F is faithful. Prove that u is an epimorphism (resp. a monomorphism) as soon as $F(u)$ is surjective (resp. injective).

Exercise 1.7. Let $\mathscr{C}$ be a category. We denote by $\mathrm{id}_{\mathscr{C}} \colon \mathscr{C} \to \mathscr{C}$ the identity functor of $\mathscr{C}$ and by $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ the set of endomorphisms of the identity functor $\mathrm{id}_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$, that is,

$$\mathrm{End}\,(\mathrm{id}_{\mathscr{C}}) = \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{C})}(\mathrm{id}_{\mathscr{C}}, \mathrm{id}_{\mathscr{C}}).$$

Prove that the composition law on $\mathrm{End}\,(\mathrm{id}_{\mathscr{C}})$ is commutative.

Chapter 2

Limits

After treating the particular cases of kernels and cokernels, products and coproducts, we shall construct limits and colimits, starting with limits in the category **Set**. We show that limits may be obtained as a combination of products and kernels, hence that colimits may be obtained as a combination of coproducts and cokernels. In particular the category **Set** of sets (in a given universe) admits limits and colimits, as well as the category $\mathrm{Mod}(A)$ of modules over a ring A. We have a glance to the particular cases of fiber products and coproducts. Then we introduce the fundamental notion of filtered colimits and cofinal functors. We show that in the category **Set**, filtered colimits commute with finite limits. Finally we have a glance to the theory of ind-objects. **Caution.** We may sometimes use the terms "projective limit" or "inductive limits" instead of "limit" or "colimit".

2.1 Products and coproducts

Let $\mathscr{C}$ be a category and consider a family $\{X_i\}_{i\in I}$ of objects of $\mathscr{C}$ indexed by a (small) set I. Consider the two functors

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \prod_i \mathrm{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.2}$$

Definition 2.1.1. (i) Assume that the functor in (2.1.1) is representable. In this case one denotes by $\prod_i X_i$ a representative and calls this object the product of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \times X_2$.

(ii) Assume that the functor in (2.1.2) is representable. In this case one denotes by $\coprod_i X_i$ a representative and calls this object the coproduct of the X_i's. In case I has two elements, say $I = \{1, 2\}$, one simply denotes this object by $X_1 \coprod X_2$ or even $X_1 \sqcup X_2$.

(iii) If for any family of objects $\{X_i\}_{i\in I}$, the product (resp. coproduct) exists, one says that the category $\mathscr{C}$ admits products (resp. coproducts) indexed by I.

(iv) If $X_i = X$ for all $i \in I$, one writes:

$$X^I := \prod_i X_i,\; X^{\coprod I} := \coprod_i X_i.$$

In case of additive categories (see below), one writes $\oplus_i X_i$ instead of $\coprod_i X_i$ and $X^{(I)}$ or $X^{\oplus I}$ instead of $X^{\coprod I}$. If $\mathscr{C} = \mathbf{Set}$, one often writes $\bigsqcup_i X_i$ instead of $\coprod_i X_i$. and $X^{\sqcup I}$ instead of $X^{\coprod I}$.

Note that the coproduct in $\mathscr{C}$ is the product in $\mathscr{C}^{\mathrm{op}}$.

By this definition, the product or the coproduct exists if and only if one has the isomorphisms, functorial with respect to $Y \in \mathscr{C}$:

$$\operatorname{Hom}_{\mathscr{C}}(Y, \prod_i X_i) \simeq \prod_i \operatorname{Hom}_{\mathscr{C}}(Y, X_i), \tag{2.1.3}$$

$$\operatorname{Hom}_{\mathscr{C}}(\coprod_i X_i, Y) \simeq \prod_i \operatorname{Hom}_{\mathscr{C}}(X_i, Y). \tag{2.1.4}$$

Assume that $\prod_i X_i$ exists. By choosing $Y = \prod_i X_i$ in (2.1.3), we get the morphisms

$$\pi_i \colon \prod_j X_j \to X_i.$$

Similarly, assume that $\coprod_i X_i$ exists. By choosing $Y = \coprod_i X_i$ in (2.1.4), we get the morphisms

$$\varepsilon_i \colon X_i \to \coprod_j X_j.$$

The isomorphism (2.1.3) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon Y \to X_i$, this family factorizes uniquely through $\prod_i X_i$. This is visualized by the diagram

$$\begin{tikzcd}
 & & X_i \\
Y \arrow[urr, "f_i"] \arrow[r, dotted] \arrow[drr, "f_j"'] & \prod_k X_k \arrow[ur, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
 & & X_j.
\end{tikzcd}$$

The isomorphism (2.1.4) may be translated as follows. Given an object Y and a family of morphisms $f_i \colon X_i \to Y$, this family factorizes uniquely through $\coprod_i X_i$. This is visualized by the diagram

$$\begin{tikzcd}
X_i \arrow[dr, "\varepsilon_i"'] \arrow[drr, "f_i"] & & \\
 & \coprod_k X_k \arrow[r, dotted] & Y. \\
X_j \arrow[ur, "\varepsilon_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

Example 2.1.2. (i) The category **Set** admits products (that is, products indexed by small sets) and the two definitions (that given in (1.1.1) and that given in Definition 2.1.1) coincide.
(ii) The category **Set** admits coproducts indexed by small sets, namely, the disjoint union.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits products, as defined in § 1.2. The category $\mathrm{Mod}(A)$ also admits coproducts, which are the direct sums defined in § 1.2. and are denoted $\bigoplus$.
(iv) Let X be a set and denote by $\mathfrak{X}$ the category of subsets of X. (The set $\mathfrak{X}$ is ordered by inclusion, hence defines a category.) For $S_1, S_2 \in \mathfrak{X}$, their product in the category $\mathfrak{X}$ is their intersection and their coproduct is their union.

Remark 2.1.3. The forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ commutes with products but does not commute with coproducts. The coproduct of two modules is not their disjoint union. That is the reason why the coproduct in the category $\mathrm{Mod}(A)$ is called the direct sum and is denoted differently, namely $\oplus$.

2.2 Kernels and cokernels

Let $\mathscr{C}$ be a category and consider two parallel arrows $f, g : X_0 \rightrightarrows X_1$ in $\mathscr{C}$. Consider the two functors

$$\mathscr{C}^{\mathrm{op}} \to \mathbf{Set}, Y \mapsto \ker(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.1}$$

$$\mathscr{C} \to \mathbf{Set}, Y \mapsto \ker(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.2}$$

Definition 2.2.1. (i) Assume that the functor in (2.2.1) is representable. In this case one denotes by $\ker(f, g)$ a representative and calls this object a kernel (one also says an equalizer) of (f, g).

(ii) Assume that the functor in (2.2.2) is representable. In this case one denotes by $\mathrm{Coker}(f, g)$ a representative and calls this object a cokernel (one also says a co-equalizer) of (f, g).

(iii) A sequence $Z \to X_0 \rightrightarrows X_1$ (resp. $X_0 \rightrightarrows X_1 \to Z$) is exact if Z is isomorphic to the kernel (resp. cokernel) of $X_0 \rightrightarrows X_1$.

(iv) Assume that the category $\mathscr{C}$ admits a zero-object 0. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. A kernel (resp. a cokernel) of f, if it exists, is a kernel (resp. a cokernel) of $f, 0\colon X \rightrightarrows Y$. It is denoted $\ker(f)$ (resp. $\mathrm{Coker}(f)$).

Note that the cokernel in $\mathscr{C}$ is the kernel in $\mathscr{C}^{\mathrm{op}}$.

By this definition, the kernel or the cokernel of $f, g\colon X_0 \rightrightarrows X_1$ exists if and only if one has the isomorphisms, functorial in $Y \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(Y, \ker(f, g)) \simeq \ker(\mathrm{Hom}_{\mathscr{C}}(Y, X_0) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(Y, X_1)), \tag{2.2.3}$$

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coker}(f, g), Y) \simeq \ker(\mathrm{Hom}_{\mathscr{C}}(X_1, Y) \rightrightarrows \mathrm{Hom}_{\mathscr{C}}(X_0, Y)). \tag{2.2.4}$$

Assume that $\ker(f, g)$ exists. By choosing $Y = \ker(f, g)$ in (2.2.3), we get the morphism

$$h\colon \ker(X_0 \rightrightarrows X_1) \to X_0.$$

Similarly, assume that $\mathrm{Coker}(f,g)$ exists. By choosing $Y = \mathrm{Coker}(f,g)$ in (2.2.4), we get the morphism

$$k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1).$$

Proposition 2.2.2. *The morphism* $h\colon \ker(X_0 \rightrightarrows X_1) \to X_0$ *is a monomorphism and the morphism* $k\colon X_1 \to \mathrm{Coker}(X_0 \rightrightarrows X_1)$ *is an epimorphism.*

Proof. (i) Set $K = \ker(X_0 \rightrightarrows X_1) \to X_0$ and consider a pair of parallel arrows $a, b\colon Y \rightrightarrows K$ such that $h \circ a = h \circ b = w$. Then $f \circ w = f \circ h \circ a = g \circ h \circ a = g \circ h \circ b = g \circ w$. Hence w factors *uniquely* through h, and this implies $a = b$.
(ii) The case of cokernels follows, by reversing the arrows. ☐

The isomorphism (2.2.3) may be translated as follows. Given an objet Y and a morphism $u\colon Y \to X_0$ such that $f \circ u = g \circ u$, the morphism u factors uniquely through $\ker(f,g)$. This is visualized by the diagram

$$\begin{tikzcd}
\ker(f,g) \arrow[r, "h"] & X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X_1. \\
& Y \arrow[u, "u"] \arrow[ul, dotted] \arrow[ur] &
\end{tikzcd} \tag{2.2.5}$$

The isomorphism (2.2.4) may be translated as follows. Given an objet Y and a morphism $v\colon X_1 \to Y$ such that $v \circ f = v \circ g$, the morphism v factors uniquely through $\mathrm{Coker}(f,g)$. This is visualized by diagram:

$$\begin{tikzcd}
X_0 \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[dr] & X_1 \arrow[r, "k"] \arrow[d, "v"] & \mathrm{Coker}(f,g). \arrow[dl, dotted] \\
& Y &
\end{tikzcd} \tag{2.2.6}$$

Example 2.2.3. (i) The category **Set** admits kernels and the two definitions (that given in (1.1.7) and that given in Definition 2.2.1) coincide.
(ii) The category **Set** admits cokernels. If $f, g\colon Z_0 \rightrightarrows Z_1$ are two maps, the cokernel of (f,g) is the quotient set $Z_1/\mathscr{R}$ where $\mathscr{R}$ is the equivalence relation generated by the relation $x \sim y$ if there exists $z \in Z_0$ with $f(z) = x$ and $g(z) = y$.
(iii) Let A be a ring. The category $\mathrm{Mod}(A)$ admits a zero object. Hence, the kernel or the cokernel of a morphism f means the kernel or the cokernel of $(f,0)$. As already mentioned, the kernel of a linear map $f\colon M \to N$ is the A-module $f^{-1}(0)$ and the cokernel is the quotient module $M/\operatorname{Im} f$. The kernel and cokernel are visualized by the diagrams

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & X_0 \arrow[r, "f"] & X_1, \\
& Y \arrow[u, "u"] \arrow[ul, dotted] \arrow[ur, "0"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[dr, "0"] & X_1 \arrow[r, "k"] \arrow[d, "v"] & \mathrm{Coker}(f). \arrow[dl, dotted] \\
& Y &
\end{tikzcd}$$

2.3 Limits

Let us generalize and unify the preceding constructions.

Definition 2.3.1. Let I and $\mathscr{C}$ categories with I small. A projective system (resp. an inductive system) in $\mathscr{C}$ indexed by I is nothing but a functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$).

For example, if $(I, \leq)$ is a pre-ordered set, I the associated category, an inductive system indexed by I is the data of a family $(X_i)_{i\in I}$ of objects of $\mathscr{C}$ and for all $i \leq j$, a morphism $X_i \to X_j$ with the natural compatibility conditions.

Projective limits in Set

Assume first that $\mathscr{C}$ is the category **Set** and let us consider a projective system $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$. One sets

$$\lim \beta = \{\{x_i\}_i \in \prod_i \beta(i); \beta(s)(x_j) = x_i \text{ for all } s \in \mathrm{Hom}_I(i,j)\}. \tag{2.3.1}$$

The next result is obvious.

Lemma 2.3.2. *Let $\beta\colon I^{\mathrm{op}} \to \mathbf{Set}$ be a functor and let $X \in \mathbf{Set}$. There is a natural isomorphism*

$$\mathrm{Hom}_{\mathbf{Set}}(X, \lim \beta) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathbf{Set}}(X, \beta),$$

where $\mathrm{Hom}_{\mathbf{Set}}(X,\beta)$ denotes the functor $I^{\mathrm{op}} \to \mathbf{Set}$, $i \mapsto \mathrm{Hom}_{\mathbf{Set}}(X, \beta(i))$.

Limits and colimits

Consider now two functors $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$. For $X \in \mathscr{C}$, we get functors from I^{op} to **Set**:

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{C}}(X,\beta)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathscr{C}}(X,\beta(i)) \in \mathbf{Set},\\ &\mathrm{Hom}_{\mathscr{C}}(\alpha,X)\colon I^{\mathrm{op}} \ni i \mapsto \mathrm{Hom}_{\mathscr{C}}(\alpha,X) \in \mathbf{Set}.\end{aligned}$$

Definition 2.3.3. (i) Assume that the functor $X \mapsto \lim \mathrm{Hom}_{\mathscr{C}}(X,\beta)$ is representable. We denote by $\lim \beta$ its representative and say that the functor β admits a limit (or "a projective limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim \beta) \simeq \lim \mathrm{Hom}_{\mathscr{C}}(X,\beta). \tag{2.3.2}$$

(ii) Assume that the functor $X \mapsto \lim \mathrm{Hom}_{\mathscr{C}}(\alpha,X)$ is representable. We denote by $\mathrm{colim}\,\alpha$ its representative and say that the functor α admits a colimit (or "an inductive limit") in $\mathscr{C}$. In other words, we have the isomorphism, functorial in $X \in \mathscr{C}$:

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{colim}\,\alpha, X) \simeq \lim \mathrm{Hom}_{\mathscr{C}}(\alpha,X), \tag{2.3.3}$$

Remark 2.3.4. The limit of the functor β is not only the object $\lim \beta$ but also the isomorphism of functors given in (2.3.2), and similarly with colimits.

When $\mathscr{C} = \mathbf{Set}$ this definition of $\lim \beta$ coincides with the former one, in view of Lemma 2.3.2.

Notice that both limits and colimits are defined using limits in **Set**.

Assume that $\lim \beta$ exists in $\mathscr{C}$. One gets:

$$\lim \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \beta) \simeq \operatorname{Hom}_{\mathscr{C}}(\lim \beta, \lim \beta)$$

and the identity of $\lim \beta$ defines a family of morphisms

$$\rho_i \colon \lim \beta \to \beta(i).$$

Consider a family of morphisms $\{f_i \colon X \to \beta(i)\}_{i \in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

$$f_j = f_i \circ f(s) \text{ for all } s \in \operatorname{Hom}_I(i, j). \tag{2.3.4}$$

This family of morphisms is nothing but an element of $\lim\limits_i \operatorname{Hom}(X, \beta(i))$, hence by (2.3.2), an element of $\operatorname{Hom}(X, \lim \beta, X)$. Therefore, $\lim \beta$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i \colon X \to \beta(i)\}_{i \in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.4), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \lim \beta. \end{cases} \tag{2.3.5}$$

This is visualized by the diagram:

$$\begin{tikzcd}
 & & \beta(i) \\
X \arrow[urr, "f_i"] \arrow[drr, "f_j"'] \arrow[r, dotted] & \lim \beta \arrow[ur, "\rho_i"'] \arrow[dr, "\rho_j"] & \\
 & & \beta(j) \arrow[uu, "\beta(s)"']
\end{tikzcd}$$

Similarly, assume that $\operatorname{colim} \alpha$ exists in $\mathscr{C}$. One gets:

$$\lim \operatorname{Hom}_{\mathscr{C}}(\alpha, \operatorname{colim} \alpha) \simeq \operatorname{Hom}_{\mathscr{C}}(\operatorname{colim} \alpha, \operatorname{colim} \alpha)$$

and the identity of $\operatorname{colim} \alpha$ defines a family of morphisms

$$\rho_i \colon \alpha(i) \to \operatorname{colim} \alpha.$$

Consider a family of morphisms $\{f_i \colon \alpha(i) \to X\}_{i \in I}$ in $\mathscr{C}$ satisfying the compatibility conditions

$$f_i = f_j \circ f(s) \text{ for all } s \in \operatorname{Hom}_I(i, j). \tag{2.3.6}$$

This family of morphisms is nothing but an element of $\lim\limits_i \operatorname{Hom}(\alpha(i), X)$, hence by (2.3.3), an element of $\operatorname{Hom}(\operatorname{colim} \alpha, X)$. Therefore, $\operatorname{colim} \alpha$ is characterized by the "universal property":

$$\begin{cases} \text{for all } X \in \mathscr{C} \text{ and all family of morphisms } \{f_i \colon \alpha(i) \to X\}_{i \in I} \\ \text{in } \mathscr{C} \text{ satisfying (2.3.6), all morphisms } f_i\text{'s factorize uniquely} \\ \text{through } \operatorname{colim} \alpha. \end{cases} \tag{2.3.7}$$

This is visualized by the diagram:

$$\begin{tikzcd}
\alpha(i) \arrow[dd, "\alpha(s)"'] \arrow[dr, "\rho_i"] \arrow[drr, "f_i"] & & \\
& \operatorname{colim}\alpha \arrow[r, dotted] & X \\
\alpha(j) \arrow[ur, "\rho_j"] \arrow[urr, "f_j"'] & &
\end{tikzcd}$$

Example 2.3.5. Let X be a set and let $\mathfrak{X}$ be the category of subsets of X (see Example 2.1.2 (iv)). Let $\beta\colon I^{\mathrm{op}} \to \mathfrak{X}$ and $\alpha\colon I \to \mathfrak{X}$ be two functors. Then

$$\lim \beta \simeq \bigcap_i \beta(i), \ \operatorname{colim} \alpha \simeq \bigcup_i \alpha(i).$$

Examples 2.3.6. (i) When the category I is discrete, limits and colimits indexed by I are nothing but products and coproducts indexed by I.
(ii) Consider the category I with two objects and two parallel morphisms other than identities, visualized by $\bullet \rightrightarrows \bullet$. A functor $\alpha\colon I \to \mathscr{C}$ is characterized by two parallel arrows in $\mathscr{C}$:

$$f, g\colon X_0 \rightrightarrows X_1 \tag{2.3.8}$$

In the sequel we shall identify such a functor with the diagram (2.3.8). Then, the kernel (resp. cokernel) of (f, g) is nothing but the limit (resp. colimit) of the functor α.
(iii) If I is the empty category and $\alpha\colon I \to \mathscr{C}$ is a functor, then $\lim \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has a terminal object $\mathrm{pt}_{\mathscr{C}}$, and in this case $\lim \alpha \simeq \mathrm{pt}_{\mathscr{C}}$. Similarly, $\operatorname{colim} \alpha$ exists in $\mathscr{C}$ if and only if $\mathscr{C}$ has an initial object $\varnothing_{\mathscr{C}}$, and in this case $\operatorname{colim} \alpha \simeq \varnothing_{\mathscr{C}}$.
(iv) If I admits a terminal object, say i_o and if $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $\alpha\colon I \to \mathscr{C}$ are functors, then

$$\lim \beta \simeq \beta(i_o), \ \operatorname{colim} \alpha \simeq \alpha(i_o).$$

This follows immediately of (2.3.7) and (2.3.5).

If every functor from I^{op} to $\mathscr{C}$ admits a limit, one says that $\mathscr{C}$ admits limits indexed by I.

Caution We shall often neglect the adjective “small” before the words “limit” and “colimit”.

Remark 2.3.7. Assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by I. Then $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ (resp. $\operatorname{colim} : \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$) is a functor.

Definition 2.3.8. One says that a category $\mathscr{C}$ admits small limits (resp. small colimits) if for any small category I and any functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) $\lim \beta$ (resp. $\operatorname{colim} \alpha$) exists in $\mathscr{C}$.

Similarly one says that $\mathscr{C}$ admits finite limits or colimits if the preceding conditions hold when assuming that I is finite.

Limits as kernels and products

We have seen that products and kernels (resp. coproducts and cokernels) are particular cases of limits (resp. colimits). One can show that conversely, limits can be obtained as kernels of products and colimits can be obtained as cokernels of coproducts.

Recall that for a category I, $\mathrm{Mor}(I)$ denote the set of morphisms in I. There are two natural maps (source and target) from $\mathrm{Mor}(I)$ to $\mathrm{Ob}(I)$:

$$\begin{aligned} \sigma\colon \mathrm{Mor}(I) \to \mathrm{Ob}(I), &\quad (s\colon i \to j) \mapsto i, \\ \tau\colon \mathrm{Mor}(I) \to \mathrm{Ob}(I), &\quad (s\colon i \to j) \mapsto j. \end{aligned}$$

Let $\mathscr{C}$ be a category which admits limits and let $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ be a functor. For $s\colon i \to j$, we get two morphisms in $\mathscr{C}$:

$$\begin{tikzcd} \beta(i) \times \beta(j) \arrow[r, shift left, "\mathrm{id}_{\beta(i)}"] \arrow[r, shift right, "\beta(s)"'] & \beta(i) \end{tikzcd}$$

from which we deduce two morphisms in $\mathscr{C}$: $\prod_{k\in I} \beta(k) \rightrightarrows \beta(\sigma(s))$. These morphisms define the two morphisms in $\mathscr{C}$:

$$\begin{tikzcd} \prod_{k\in I} \beta(k) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod_{s\in \mathrm{Mor}(I)} \beta(\sigma(s)). \end{tikzcd} \tag{2.3.9}$$

Similarly, assume that $\mathscr{C}$ admits colimits and let $\alpha\colon I \to \mathscr{C}$ be a functor. By reversing the arrows, one gets the two morphisms in $\mathscr{C}$:

$$\begin{tikzcd} \coprod_{s\in \mathrm{Mor}(I)} \alpha(\sigma(s)) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \coprod_{k\in I} \alpha(k). \end{tikzcd} \tag{2.3.10}$$

Proposition 2.3.9. (i) $\lim \beta$ *is the kernel of* (a, b) *in* (2.3.9),

(ii) $\mathrm{colim}\, \alpha$ *is the cokernel of* (a, b) *in* (2.3.10).

Sketch of proof. By the definition of limits and colimits we are reduced to check (i) when $\mathscr{C} = \mathbf{Set}$ and in this case this is obvious. ☐

In particular, a category $\mathscr{C}$ admits finite limits if and only if it satisfies:

(i) $\mathscr{C}$ admits a terminal object,

(ii) for any $X, Y \in \mathrm{Ob}(\mathscr{C})$, the product $X \times Y$ exists in $\mathscr{C}$,

(iii) for any parallel arrows in $\mathscr{C}$, $f, g\colon X \rightrightarrows Y$, the kernel exists in $\mathscr{C}$.

There is a similar result for finite colimits, replacing a terminal object by an initial object, products by coproducts and kernels by cokernels.

Theorem 2.3.10. (a) *The category* **Set** *admits small limits and colimits.*

(b) *Let A be a ring. The category* $\mathrm{Mod}(A)$ *admits small limits and colimits and the forgetful functor for*$\mathrm{Mod}(A) \to$ **Set** *commutes with limits.*

Proof. (i) Both categories admit small products and coproducts as well as kernels and cokernels (see Example 2.2.3).

(ii) The forgetful functor *for* commutes with products and kernels. ☐

Recall that the forgetful functor *for* does not commute with coproducts (see Remark 2.1.3).

2.4 Fiber products and coproducts

Consider the category I with three objects $\{a,b,c\}$ and two morphisms other than the identities, visualized by the diagram

$$a \leftarrow c \rightarrow b.$$

Let $\mathscr{C}$ be a category. A functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ (resp. $\alpha\colon I \to \mathscr{C}$) is nothing but the data of three objects X_0, X_1, Y and two morphisms (f,g) (resp. (k,l)) visualized by the arrows on the left (resp. on the right)

$$X_0 \xrightarrow{f} Y \xleftarrow{g} X_1,\ X_0 \xleftarrow{k} W \xrightarrow{l} X_1.$$

The fiber product $X_0 \times_Y X_1$ of X_0 and X_1 over Y, if it exists, is the limit of β.

The fiber coproduct $X_0 \sqcup_W X_1$ of X_0 and X_1 over W, if it exists, is the colimit of α.

Consider a commutative diagram in $\mathscr{C}$:

(2.4.1)
$$\begin{tikzcd} W \arrow[r] \arrow[d] & X_1 \arrow[d] \\ X_0 \arrow[r] & Y \end{tikzcd}$$

Definition 2.4.1. The square (2.4.1) is Cartesian if $W \simeq X_0 \times_Y X_1$. It is co-Cartesian if $Y \simeq X_0 \sqcup_W X_1$

Proposition 2.4.2. (a) *Assume that $\mathscr{C}$ admits products of two objects and kernels. Then $X_0 \times_Y X_1$ is isomorphic to $\ker(f,g)$, the equalizer of $(f,g)\colon X_0 \times X_1 \rightrightarrows Y$.*

(b) *Assume that $\mathscr{C}$ admits coproducts of two objects and cokernels. Then $X_0 \sqcup_W X_1$ is isomorphic to $\mathrm{Coker}(k,l)$, the co-equalizer of $(k,l)\colon W \rightrightarrows X_0 \sqcup X_1$.*

Proof. It follows from the characterizations of limits and colimits given in (2.3.5) and (2.3.7). ☐

Proposition 2.4.3. (a) *The category $\mathscr{C}$ admits finite limits if and only if it admits fiber products and a terminal object.*

(b) *The category $\mathscr{C}$ admits finite colimits if and only if it admits fiber coproducts and an initial object.*

Proof. (a) If $\mathscr{C}$ admits finite limits, then it admits fiber products by Proposition 2.4.2 (a). Conversely, if $\mathscr{C}$ admits a terminal object $\mathrm{pt}_{\mathscr{C}}$ and fiber products, then it admits product of two objects (X_0, X_1), namely $X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1$. It admits kernels since given $(f,g)\colon X \rightrightarrows Y$, then $\ker(f,g) \simeq X \times_Y X$ again by Proposition 2.4.2 (a).
(b) is deduced from (a) by reversing the arrows. ☐

To summarize, assuming that $\mathscr{C}$ admits finite limits and colimits, we have for $f\colon X_0 \to Y$, $g\colon X_1 \to Y$ and when $X = X_0 = X_1$

(2.4.2) $$X_0 \times_Y X_1 \simeq \ker(X_0 \times X_1 \rightrightarrows Y),\ \ker(f,g) \simeq X \times_Y X,$$

and for $k\colon W \to X_0$, $l\colon W \to X_1$ and when $X = X_0 = X_1$

$$X_0 \sqcup_W X_1 \simeq \operatorname{Coker}(W \rightrightarrows X_0 \times X_1),\ \operatorname{Coker}(k,l) \simeq X \sqcup_W X. \tag{2.4.3}$$

Moreover

$$X_0 \times X_1 \simeq X_0 \times_{\mathrm{pt}_{\mathscr{C}}} X_1,\ X_0 \sqcup X_1 \simeq X_0 \sqcup_{\varnothing_{\mathscr{C}}} X_1 \tag{2.4.4}$$

Definition 2.4.4. Let $\mathscr{C}$ be a category which admits finite limits and colimits and let $f\colon X \to Y$ be a morphism. One sets

$$\operatorname{Coim} f := \operatorname{Coker}(X \times_Y X \rightrightarrows X),\ \operatorname{Im} f := \ker(Y \rightrightarrows Y \sqcup_X Y). \tag{2.4.5}$$

Here, the fiber product $X \times_Y X$ as well as the fiber coproduct $Y \sqcup_X Y$ are associated with two copies of the map f.

One calls $\operatorname{Coim}(f)$ and $\operatorname{Im}(f)$ the co-image and the image of f, respectively.

One has a natural epimorphism $s\colon X \to \operatorname{Coim} f$ and a natural monomorphism $t\colon \operatorname{Im} f \to Y$. Moreover, one can construct a natural morphism $u\colon \operatorname{Coim}(f) \to \operatorname{Im}(f)$ such that the composition $X \to \operatorname{Coim}(f) \to \operatorname{Im}(f) \to Y$ is f (see [KS06, Prop. 5.1.2] and Section 5.1 for a similar construction in the abelian setting).

2.5 Properties of limits

Double limits

For two categories I and $\mathscr{C}$, recall the notation $\mathscr{C}^I := \operatorname{Fct}(I, \mathscr{C})$ and for a third category J, recall the equivalence (1.3.4);

$$\operatorname{Fct}(I \times J, \mathscr{C}) \simeq \operatorname{Fct}(I, \operatorname{Fct}(J, \mathscr{C})).$$

Consider a bifunctor $\beta\colon I^{\mathrm{op}} \times J^{\mathrm{op}} \to \mathscr{C}$ with I and J small. It defines a functor $\beta_J\colon I^{\mathrm{op}} \to \mathscr{C}^{J^{\mathrm{op}}}$ as well as a functor $\beta_I\colon J^{\mathrm{op}} \to \mathscr{C}^{I^{\mathrm{op}}}$. One easily checks that

$$\lim \beta \simeq \lim \lim \beta_J \simeq \lim \lim \beta_I. \tag{2.5.1}$$

Similarly, if $\alpha\colon I \times J \to \mathscr{C}$ is a bifunctor, it defines a functor $\alpha_J\colon I \to \mathscr{C}^J$ as well as a functor $\alpha_I\colon J \to \mathscr{C}^I$ and one has the isomorphisms

$$\operatorname{colim} \alpha \simeq \operatorname{colim}(\operatorname{colim} \alpha_J) \simeq \operatorname{colim}(\operatorname{colim} \alpha_I). \tag{2.5.2}$$

In other words:

$$\lim_{i,j} \beta(i,j) \simeq \lim_j \lim_i (\beta(i,j)) \simeq \lim_i \lim_j (\beta(i,j)), \tag{2.5.3}$$

$$\operatorname*{colim}_{i,j} \alpha(i,j) \simeq \operatorname*{colim}_j (\operatorname*{colim}_i (\alpha(i,j)) \simeq \operatorname*{colim}_i \operatorname*{colim}_j (\alpha(i,j)). \tag{2.5.4}$$

Limits with values in a category of functors

Consider another category $\mathscr{A}$ and a functor $\beta\colon I^{\mathrm{op}} \to \mathrm{Fct}(\mathscr{A},\mathscr{C})$. It defines a functor $\widetilde{\beta}\colon I^{\mathrm{op}} \times \mathscr{A} \to \mathscr{C}$, hence for each $A \in \mathscr{A}$, a functor $\widetilde{\beta}(A)\colon I^{\mathrm{op}} \to \mathscr{C}$. Assuming that $\mathscr{C}$ admits limits indexed by I, one checks easily that $A \mapsto \lim \widetilde{\beta}(A)$ is a functor, that is, an object of $\mathrm{Fct}(\mathscr{A},\mathscr{C})$, and is a limit of β. There is a similar result for colimits. Hence:

Proposition 2.5.1. *Let I be a small category and assume that $\mathscr{C}$ admits limits indexed by I. Then for any category $\mathscr{A}$, the category $\mathrm{Fct}(\mathscr{A},\mathscr{C})$ admits limits indexed by I. Moreover, if $\beta\colon I^{\mathrm{op}} \to \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\lim \beta \in \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by*

$$(\lim \beta)(A) = \lim\,(\beta(A)), \quad A \in \mathscr{A}.$$

Similarly, assume that $\mathscr{C}$ admits colimits indexed by I. Then for any category $\mathscr{A}$, the category $\mathrm{Fct}(\mathscr{A},\mathscr{C})$ admits colimits indexed by I. Moreover, if $\alpha\colon I \to \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is a functor, then $\operatorname{colim} \alpha \in \mathrm{Fct}(\mathscr{A},\mathscr{C})$ is given by

$$(\operatorname{colim} \alpha)(A) = \operatorname{colim}\,(\alpha(A)), \quad A \in \mathscr{A}.$$

Corollary 2.5.2. *Let $\mathscr{C}$ be a category. Then the categories $\mathscr{C}^\wedge$ and $\mathscr{C}^\vee$ admit small limits and colimits.*

Composition of limits

Let $I, \mathscr{C}$ and $\mathscr{C}'$ be categories with I small and let $\alpha\colon I \to \mathscr{C}$, $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{C}'$ be functors. When $\mathscr{C}$ and $\mathscr{C}'$ admit limits or colimits indexed by I, there are natural morphisms

$$F(\lim \beta) \to \lim\,(F \circ \beta), \tag{2.5.5}$$

$$\operatorname{colim}\,(F \circ \alpha) \to F(\operatorname{colim} \alpha). \tag{2.5.6}$$

This follows immediately from (2.3.7) and (2.3.5).

Definition 2.5.3. Let I be a small category and let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor.

(i) Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit limits indexed by I. One says that F commutes with such limits if (2.5.5) is an isomorphism.

(ii) Similarly, assume that $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I. One says that F commutes with such colimits if (2.5.6) is an isomorphism.

Examples 2.5.4. (i) Let $\mathscr{C}$ be a category which admits limits indexed by I and let $X \in \mathscr{C}$. By (2.3.2), the functor $\mathrm{Hom}_{\mathscr{C}}(X, \bullet)\colon \mathscr{C} \to \mathbf{Set}$ commutes with limits indexed by I. Similarly, if $\mathscr{C}$ admits colimits indexed by I, then the functor $\mathrm{Hom}_{\mathscr{C}}(\bullet, X)\colon \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ commutes with limits indexed by I, by (2.3.3).
(ii) Let I and J be two small categories and assume that $\mathscr{C}$ admits limits (resp. colimits) indexed by $I \times J$. Then the functor $\lim\colon \mathrm{Fct}(J^{\mathrm{op}},\mathscr{C}) \to \mathscr{C}$ (resp. $\operatorname{colim}\colon \mathrm{Fct}(J,\mathscr{C}) \to \mathscr{C}$) commutes with limits (resp. colimits) indexed by I. This follows from the isomorphisms (2.5.1) and (2.5.2).
(iii) Let $\mathbf{k}$ be a field, $\mathscr{C} = \mathscr{C}' = \mathrm{Mod}(\mathbf{k})$, and let $X \in \mathscr{C}$. Then the functor $\mathrm{Hom}_{\mathbf{k}}(X, \bullet)$ does not commute with colimit if X is infinite dimensional.

Proposition 2.5.5. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor and let I be a small category.*

(i) *Assume that $\mathscr{C}$ and $\mathscr{C}'$ admit limits indexed by I and F admits a left adjoint $G\colon \mathscr{C}' \to \mathscr{C}$. Then F commutes with limits indexed by I, that is, $F(\varprojlim_i \beta(i)) \simeq \varprojlim_i F(\beta(i))$.*

(ii) *Similarly, if $\mathscr{C}$ and $\mathscr{C}'$ admit colimits indexed by I and F admits a right adjoint, then F commutes with such colimits.*

Proof. It is enough to prove the first assertion. To check that (2.5.5) is an isomorphism, we apply Corollary 1.4.4. Let $Y \in \mathscr{C}'$. One has the chain of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathscr{C}'}(Y, F(\varprojlim_i \beta(i))) &\simeq \mathrm{Hom}_{\mathscr{C}}(G(Y), \varprojlim_i \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathscr{C}}(G(Y), \beta(i)) \\
&\simeq \varprojlim_i \mathrm{Hom}_{\mathscr{C}'}(Y, F(\beta(i))) \\
&\simeq \mathrm{Hom}_{\mathscr{C}'^\wedge}(Y, \varprojlim_i F(\beta(i))).
\end{aligned}$$

□

2.6 Filtered colimits

As already seen in Theorem 2.3.10, the category **Set** admits small colimits. In the category **Set** one uses the notation $\bigsqcup$ rather than $\coprod$.

We shall construct colimits more explicitly.

Let $\alpha\colon I \to$ **Set** be a functor (with I small) and consider the relation on $\bigsqcup_{i\in I} \alpha(i))$:

$$\begin{cases} \alpha(i) \ni x \mathscr{R} y \in \alpha(j) \text{ if there exists } k \in I,\ s\colon i \to k \text{ and } t\colon j \to k \\ \text{with } \alpha(s)(x) = \alpha(t)(y). \end{cases} \tag{2.6.1}$$

The relation $\mathscr{R}$ is reflexive and symmetric but is not transitive in general.

Proposition 2.6.1. *With the notations above, denote by $\sim$ the equivalence relation generated by $\mathscr{R}$. Then*

$$\mathrm{colim}\, \alpha \simeq (\bigsqcup_{i\in I} \alpha(i))/\sim .$$

Proof. Apply Proposition 2.3.9 and Example 2.2.3 (ii). □

For a ring A, the category $\mathrm{Mod}(A)$ admits coproducts and cokernels. Hence, the category $\mathrm{Mod}(A)$ admits colimits. One shall be aware that the functor *for*: $\mathrm{Mod}(A) \to$ **Set** does not commute with colimits. For example, if I is empty and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then $\alpha(I) = \{0\}$ and *for*$(\{0\})$ is not an initial object in **Set**.

Definition 2.6.2. A category I is called filtered if it satisfies the conditions (i)–(iii) below.

(i) I is non empty,

(ii) for any i and j in I, there exists $k \in I$ and morphisms $i \to k, j \to k$,

(iii) for any parallel morphisms $f, g\colon i \rightrightarrows j$, there exists a morphism $h\colon j \to k$ such that $h \circ f = h \circ g$.

One says that I is cofiltered if I^{op} is filtered.

The conditions (ii)–(iii) of being filtered are visualized by the diagrams:

$$\begin{tikzcd}
i \arrow[dr, dotted] & \\
& k \\
j \arrow[ur, dotted] &
\end{tikzcd}
\qquad
\begin{tikzcd}
i \arrow[r, shift left] \arrow[r, shift right] \arrow[dr, dotted] & j \arrow[d, dotted] \\
& k
\end{tikzcd}$$

Of course, if $(I, \leq)$ is a non-empty directed ordered set, then the associated category I is filtered.

Proposition 2.6.3. *Let $\alpha\colon I \to$ **Set** be a functor, with I filtered. The relation $\mathscr{R}$ given in (2.6.1) on $\coprod_i \alpha(i)$ is an equivalence relation.*

Proof. Let $x_j \in \alpha(i_j)$, $j = 1, 2, 3$ with $x_1 \sim x_2$ and $x_2 \sim x_3$. There exist morphisms visualized by the diagram:

$$\begin{tikzcd}
i_1 \arrow[r, "s_1"] & j_1 \arrow[dr, "u_1"'] & & \\
i_2 \arrow[ur, "s_2"'] \arrow[dr, "t_2"] & & k_1 \arrow[r, "v"] & l \\
i_3 \arrow[r, "t_3"] & j_2 \arrow[ur, "u_2"] & &
\end{tikzcd}$$

such that $\alpha(s_1)x_1 = \alpha(s_2)x_2$, $\alpha(t_2)x_2 = \alpha(t_3)x_3$, and $v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$. Set $w_1 = v \circ u_1 \circ s_1$, $w_2 = v \circ u_1 \circ s_2 = v \circ u_2 \circ t_2$ and $w_3 = v \circ u_2 \circ t_3$. Then $\alpha(w_1)x_1 = \alpha(w_2)x_2 = \alpha(w_3)x_3$. Hence $x_1 \sim x_3$. ☐

Corollary 2.6.4. *Let $\alpha\colon I \to$ **Set** be a functor, with I small and filtered.*

(i) *Let S be a finite subset in $\operatorname{colim}\alpha$. Then there exists $i \in I$ such that S is contained in the image of $\alpha(i)$ by the natural map $\alpha(i) \to \operatorname{colim}\alpha$.*

(ii) *Let $i \in I$ and let x and y be elements of $\alpha(i)$ with the same image in $\operatorname{colim}\alpha$. Then there exists $s\colon i \to j$ such that $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$.*

Proof. (i) Denote by $\lambda\colon \bigsqcup_{i\in I} \alpha(i) \to \operatorname{colim}\alpha$ the quotient map. Let $S = \{x_1, \dots, x_n\}$. For $j = 1, \dots, n$, there exists $y_j \in \alpha(i_j)$ such that $x_j = \lambda(y_j)$. Choose $k \in I$ such that there exist morphisms $s_j\colon \alpha(i_j) \to \alpha(k)$. Then $x_j = \lambda(\alpha(s_j(y_j)))$.
(ii) For $x, y \in \alpha(i)$, $x\mathscr{R}y$ if and only if there exists $s\colon i \to j$ with $\alpha(s)(x) = \alpha(s)(y)$ in $\alpha(j)$. ☐

Corollary 2.6.5. *Let A be a ring and denote by for the forgetful functor* $\mathrm{Mod}(A) \to$ **Set**. *Then the functor for commutes with filtered colimits. In other words, if I is small and filtered and $\alpha\colon I \to \mathrm{Mod}(A)$ is a functor, then*

$$for \circ (\operatorname*{colim}_i \alpha(i)) = \operatorname*{colim}_i (for \circ \alpha(i)).$$

The proof is left as an exercise (see Exercise 2.8).

Colimits with values in **Set** indexed by small filtered categories commute with finite limits. More precisely:

Theorem 2.6.6. *For a small filtered category I, a finite category J and a functor $\alpha\colon I \times J^{\mathrm{op}} \to \mathbf{Set}$, one has $\operatorname*{colim}_i \lim_j \alpha(i,j) \xrightarrow{\sim} \lim_j \operatorname*{colim}_i \alpha(i,j)$. In other words, the functor*

$$\operatorname{colim}\colon \operatorname{Fct}(I, \mathbf{Set}) \to \mathbf{Set}$$

commutes with finite limits.

Proof. It is enough to prove that colim commutes with kernels and with finite products.

(i) colim commutes with kernels. Let $\alpha, \beta\colon I \to \mathbf{Set}$ be two functors and let $f, g\colon \alpha \rightrightarrows \beta$ be two morphisms of functors. Define γ as the kernel of (f, g), that is, we have exact sequences

$$\gamma(i) \to \alpha(i) \rightrightarrows \beta(i).$$

Let Z denote the kernel of $\operatorname*{colim}_i \alpha(i) \rightrightarrows \operatorname*{colim}_i \beta(i)$. We have to prove that the natural map $\lambda\colon \operatorname*{colim}_i \gamma(i) \to Z$ is bijective.

(i) (a) The map λ is surjective. Indeed for $x \in Z$, represent x by some $x_i \in \alpha(i)$. Then $f_i(x_i)$ and $g_i(x_i)$ in $\beta(i)$ having the same image in $\operatorname{colim}\beta$, there exists $s\colon i \to j$ such that $\beta(s)f_i(x_i) = \beta(s)g_i(x_i)$. Set $x_j = \alpha(s)x_i$. Then $f_j(x_j) = g_j(x_j)$, which means that $x_j \in \gamma(j)$. Clearly, $\lambda(x_j) = x$.

(i) (b) The map λ is injective. Indeed, let $x, y \in \operatorname{colim}\gamma$ with $\lambda(x) = \lambda(y)$. We may represent x and y by elements x_i and y_i of $\gamma(i)$ for some $i \in I$. Since x_i and y_i have the same image in $\operatorname{colim}\alpha$, there exists $i \to j$ such that they have the same image in $\alpha(j)$. Therefore their images in $\gamma(j)$ will be the same.

(ii) colim commutes with finite products. The proof is similar to the preceding one and left to the reader. □

Corollary 2.6.7. *Let A be a ring and let I be a small filtered category. Then the functor* $\operatorname{colim}\colon \operatorname{Fct}(I, \operatorname{Mod}(A)) \to \operatorname{Mod}(A)$ *commutes with finite limits.*

Exact functors

Definition 2.6.8. Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a functor. Assume that both $\mathscr{C}$ and $\mathscr{C}'$ admit finite limits (resp. colimits) and F commutes with such limits. In this case one says that F is left (right) exact.

In case F is both left and right exact, one says that F is exact.

For example, if $\mathscr{C}$ admits limits indexed by a category I, the functor $\lim\colon \operatorname{Fct}(I, \mathscr{C}^{\mathrm{op}}) \to \mathscr{C}$ is left exact and similarly for the functor colim. Moreover, Theorem 2.6.6 and Corollary 2.6.7 may be translated by saying that in these situations, the functor colim is left exact.

We shall study left/right exact functors with great details in Chapter 9.

Cofinal functors

Let $\varphi\colon J \to I$ be a functor. If there are no risk of confusion, we still denote by φ the associated functor $\varphi\colon J^{\mathrm{op}} \to I^{\mathrm{op}}$. For two functors $\alpha\colon I \to \mathscr{C}$ and $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$, we have natural morphisms:

$$\lim\,(\beta \circ \varphi) \leftarrow \lim \beta, \tag{2.6.2}$$

$$\operatorname{colim}\,(\alpha \circ \varphi) \to \operatorname{colim} \alpha. \tag{2.6.3}$$

This follows immediately of (2.3.7) and (2.3.5).

Definition 2.6.9. (a) Let $\varphi\colon J \to I$ be a functor. Assume that φ is fully faithful and I is filtered. One says that φ is cofinal if for any $i \in I$ there exists $j \in J$ and a morphism $s\colon i \to \varphi(j)$.

(b) Let I be a filtered category. One says that I is cofinally small if there exists a fully faithful functor $\varphi\colon J \to I$ such that J is small and φ is cofinal.

Example 2.6.10. A subset $J \subset \mathbb{N}$ defines a cofinal subcategory of $(\mathbb{N}, \leq)$ if and only if it is infinite.

Proposition 2.6.11. *Let $\varphi\colon J \to I$ be a fully faithful functor. Assume that I is filtered and φ is cofinal. Then*

(i) *for any category $\mathscr{C}$ and any functor $\beta\colon I^{\mathrm{op}} \to \mathscr{C}$, the morphism (2.6.2) is an isomorphism,*

(ii) *for any category $\mathscr{C}$ and any functor $\alpha\colon I \to \mathscr{C}$, the morphism (2.6.3) is an isomorphism.*

Proof. Let us prove (ii), the other proof being similar. By the hypothesis, for each $i \in I$ we get a morphism $\alpha(i) \to \operatorname*{colim}_{j\in J}(\alpha \circ \varphi(j))$ from which one deduce a morphism

$$\operatorname*{colim}_{i\in I} \alpha(i) \to \operatorname*{colim}_{j\in J}(\alpha \circ \varphi(j)).$$

One checks easily that this morphism is inverse to the morphism in (2.5.6). □

Example 2.6.12. Let X be a topological space, $x \in X$ and denote by I_x the set of open neighborhoods of x in X. We endow I_x with the order: $U \leq V$ if $V \subset U$. Given U and V in I_x, and setting $W = U \cap V$, we have $U \leq W$ and $V \leq W$. Therefore, I_x is filtered.

Denote by $\mathscr{C}^0(U)$ the $\mathbb{C}$-vector space of complex valued continuous functions on U. The restriction maps $\mathscr{C}^0(U) \to \mathscr{C}^0(V), V \subset U$ define an inductive system of $\mathbb{C}$-vector spaces indexed by I_x. One sets

$$\mathscr{C}^0_{X,x} = \operatorname*{colim}_{U\in I_x} \mathscr{C}^0(U). \tag{2.6.4}$$

An element φ of $\mathscr{C}^0_{X,x}$ is called a germ of continuous function at 0. Such a germ is an equivalence class $(U, \varphi_U)/\sim$ with U a neighborhood of x, φ_U a continuous function on U, and $(U, \varphi_U) \sim 0$ if there exists a neighborhood V of x with $V \subset U$ such that the restriction of φ_U to V is the zero function. Hence, a germ of function is zero at x if this function is identically zero in a neighborhood of x.

Ind-objects

The aim of this subsection is to have a glance to the notion of ind-objects. Since we shall not use this theory in this book, we shall be extremely sketchy.

By Theorem 2.3.10, the category **Set** admits small limits and colimits. It follows from Proposition 2.5.1 that for any category $\mathscr{C}$, the big category $\mathscr{C}^\wedge$ also admits small limits and colimits. One denotes by "colim" the colimit in $\mathscr{C}^\wedge$.

One could also define "lim" in $\mathscr{C}^\vee$ but we shall concentrate here on colimits.

In the sequel we identify $\mathscr{C}$ to a full subcategory of $\mathscr{C}^\wedge$ by the Yoneda functor $\mathrm{h}_\mathscr{C}$ and when there is no risk of confusion, we shall write X instead of $\mathrm{h}_\mathscr{C}(X)$. Hence, for a small a category I and a functors $\alpha\colon I \to \mathscr{C}$, we have:

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(X, \text{“colim”}\,\alpha) \simeq \mathrm{colim}\,\mathrm{Hom}_\mathscr{C}(X, \alpha).$$

Let $A \in \mathscr{C}^\wedge$. Applying Definition 1.4.5 to the Yoneda functor, we get the category $\mathscr{C}_A$.

Lemma 2.6.13. *Let $A \in \mathscr{C}^\wedge$. Then $A \simeq \underset{(X\to A)\in\mathscr{C}_A}{\text{“colim”}} X$.*

Proof. Let $B \in \mathscr{C}^\wedge$. The maps $X \to A$ define the map

$$\mathrm{Hom}_{\mathscr{C}^\wedge}(A, B) \to \lim_{(X\to A)\in\mathscr{C}_A} \mathrm{Hom}_{\mathscr{C}^\wedge}(X, B) \simeq \lim_{(X\to A)\in\mathscr{C}_A} B(X).$$

Hence, this map is bijective by the definition of a morphism of functors. □

Definition 2.6.14. One denotes by $\mathrm{Ind}(\mathscr{C})$ the full subcategory of $\mathscr{C}^\wedge$ consisting of objects isomorphic to "colim" α for some functor $\alpha\colon I \to \mathscr{C}$ with I small and filtered. One calls an object of $\mathrm{Ind}(\mathscr{C})$ an ind-object.

The next result is important. However, since we shall not se it here, we skip the proof, referring to [KS06, Prop. 6.1.5, Th. 6.1.8]. Note that the "if"part of the first statement follows immediately from Lemma 2.6.13.

Proposition 2.6.15. (a) *Let $A \in \mathscr{C}^\wedge$. Then $A \in \mathrm{Ind}(\mathscr{C})$ if and only if the category $\mathscr{C}_A$ is filtered and cofinally small.*

(b) *The category $\mathrm{Ind}(\mathscr{C})$ admits small filtered colimits and the embedding $\mathrm{Ind}(\mathscr{C}) \hookrightarrow \mathscr{C}^\wedge$ commutes with colimits.*

The colimit in $\mathrm{Ind}(\mathscr{C})$ is denoted "colim" .

A set theoretical remark

Remark 2.6.16. As already mentioned, all categories $\mathscr{C}$, $\mathscr{C}'$ etc. belong to a given universe $\mathscr{U}$ and all limits or colimits are indexed by $\mathscr{U}$-small categories I, J, etc. Let us give an example which shows that without some care, we may have troubles.

Let $\mathscr{C}$ be a category which admits products and assume there exist $X, Y \in \mathscr{C}$ such that $\mathrm{Hom}_\mathscr{C}(X, Y)$ has more than one element. Set $M = \mathrm{Mor}(\mathscr{C})$, where $\mathrm{Mor}(\mathscr{C})$ denotes the big set of all morphisms in $\mathscr{C}$, and let $\pi = \mathrm{card}(M)$, the cardinal of the set M. We have

$$\mathrm{Hom}_\mathscr{C}(X, Y^M) \simeq \mathrm{Hom}_\mathscr{C}(X, Y)^M$$

and therefore $\mathrm{card}(\mathrm{Hom}_{\mathscr{C}}(X, Y^M) \geq 2^\pi$. On the other hand, $\mathrm{Hom}_{\mathscr{C}}(X, Y^M) \subset \mathrm{Mor}(\mathscr{C})$ which implies $\mathrm{card}(\mathrm{Hom}_{\mathscr{C}}(X, Y^M) \leq \pi$.

The "contradiction" comes from the fact that $\mathscr{C}$ does not admit products indexed by such a big set as $\mathrm{Mor}(\mathscr{C})$. (This remark is extracted from [Fre64].)

Exercises to Chapter 2

Exercise 2.1. (i) Let I be a small set and $\{X_i\}_{i\in I}$ a family of sets indexed by I. Show that $\coprod_i X_i = \bigsqcup_i X_i$, the disjoint union of the sets X_i.
(ii) Construct the natural map $\bigsqcup_i \mathrm{Hom}_{\mathbf{Set}}(Y, X_i) \to \mathrm{Hom}_{\mathbf{Set}}(Y, \bigsqcup_i X_i)$ and prove it is injective and not surjective in general.

Exercise 2.2. Let $X, Y \in \mathscr{C}$ and consider the category $\mathscr{D}$ whose objects are triplets $Z \in \mathscr{C}, f\colon Z \to X, g\colon Z \to Y$, the morphisms being the natural ones. Prove that this category admits a terminal object if and only if the product $X \times Y$ exists in $\mathscr{C}$, and that in such a case this terminal object is isomorphic to $X \times Y, X \times Y \to X, X \times Y \to Y$. Deduce that if $X \times Y$ exists, it is unique up to unique isomorphism.

Exercise 2.3. Let I and $\mathscr{C}$ be two categories with I small and denote by Δ the functor from $\mathscr{C}$ to $\mathscr{C}^I$ which, to $X \in \mathscr{C}$, associates the constant functor $\Delta(X)\colon I \ni i \mapsto X \in \mathscr{C}$, $(i \to j) \in \mathrm{Mor}(I) \mapsto \mathrm{id}_X$. Assume that any functor from I to $\mathscr{C}$ admits a colimit.
(i) Prove the formula (for $\alpha\colon I \to \mathscr{C}$ and $Y \in \mathscr{C}$):

$$\mathrm{Hom}_{\mathscr{C}}(\operatorname*{colim}_i \alpha(i), Y) \simeq \mathrm{Hom}_{\mathrm{Fct}(I,\mathscr{C})}(\alpha, \Delta(Y)).$$

(ii) Replacing I with the opposite category, deduce the formula (assuming limits exist):

$$\mathrm{Hom}_{\mathscr{C}}(X, \lim_i G(i)) \simeq \mathrm{Hom}_{\mathrm{Fct}(I^{\mathrm{op}},\mathscr{C})}(\Delta(X), G).$$

Exercise 2.4. Let $\mathscr{C}$ be a category which admits small filtered colimits. One says that an object X of $\mathscr{C}$ is of finite type if for any functor $\alpha\colon I \to \mathscr{C}$ with I filtered, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is injective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.
(Hint: let $X \in \mathrm{Mod}(A)$. To prove that if X is of finite type in the categorical sense then it is of finite type in the usual sense, use the fact that, denoting by $\mathscr{S}$ be the family of submodules of finite type of X ordered by inclusion, we have $\operatorname*{colim}_{V\in\mathscr{S}} X/V \simeq 0$.)

Exercise 2.5. Let $\mathscr{C}$ be a category which admits small filtered colimits. One says that an object X of $\mathscr{C}$ is of finite presentation if for any functor $\alpha\colon I \to \mathscr{C}$ with I small and filtered, the natural map $\operatorname{colim} \mathrm{Hom}_{\mathscr{C}}(X, \alpha) \to \mathrm{Hom}_{\mathscr{C}}(X, \operatorname{colim} \alpha)$ is bijective. Show that this definition coincides with the classical one when $\mathscr{C} = \mathrm{Mod}(A)$, for a ring A.

Exercise 2.6. In the situation of Definition 2.4.4, construct the natural morphism $u\colon \mathrm{Coim}(f) \to \mathrm{Im}(f)$ such that the composition $X \to \mathrm{Coim}(f) \to \mathrm{Im}(f) \to Y$ is f. (See [KS06, Prop. 5.1.2].)

Exercise 2.7. Let I be a filtered ordered set and let $\{A_i\}_{i\in I}$ be an inductive system of rings indexed by I.
(i) Prove that $A := \operatorname*{colim}_i A_i$ is naturally endowed with a ring structure.
(ii) Define the notion of an inductive system M_i of A_i-modules, and define the A-module $\operatorname*{colim}_i M_i$.
(iii) Let N_i (resp. M_i) be an inductive system of right (resp. left) A_i modules. Prove the isomorphism

$$\operatorname*{colim}_i (N_i \otimes_{A_i} M_i) \xrightarrow{\sim} \operatorname*{colim}_i N_i \otimes_A \operatorname*{colim}_i M_i.$$

Exercise 2.8. Prove Corollary 2.6.5.

Exercise 2.9. (i) Let $\mathscr{C}$ be a category which admits colimits indexed by a category I. Let $\alpha\colon I \to \mathscr{C}$ be a functor and let $X \in \mathscr{C}$. Construct the natural morhism

$$\operatorname*{colim}_i \operatorname{Hom}_{\mathscr{C}}(X, \alpha(i)) \to \operatorname{Hom}_{\mathscr{C}}(X, \operatorname*{colim}_i \alpha(i)). \tag{2.6.5}$$

(ii) Let $\mathbf{k}$ be a field and denote by $\mathbf{k}[x]^{\leq n}$ the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. Prove the isomorphism $\mathbf{k}[x] = \operatorname*{colim}_n \mathbf{k}[x]^{\leq n}$ and, noticing that $\mathrm{id}_{\mathbf{k}[x]} \notin \operatorname*{colim}_n \operatorname{Hom}_{\mathbf{k}}(\mathbf{k}[x], \mathbf{k}[x]^{\leq n})$, deduce that the morphism (2.6.5) is not an isomorphism in general.

Exercise 2.10. Let $\mathscr{C}$ be a category and recall (Proposition 2.5.1) that the category $\mathscr{C}^\wedge$ admits small colimits. One denotes by "colim" the colimit in $\mathscr{C}^\wedge$. By its definition, for $X \in \mathscr{C}$,

$$\operatorname{Hom}_{\mathscr{C}^\wedge}(X, \text{“colim”}\, \alpha) \simeq \operatorname{colim} \alpha \operatorname{Hom}(X, \alpha).$$

and one denotes by $\operatorname{Ind}(\mathscr{C})$ the full subcategory of $\mathscr{C}^\wedge$ consisting of objects which are small filtered colimits of objects of $\mathscr{C}$. In other words, $A \in \operatorname{Ind}(\mathscr{C})$ if there exists a small filtered category I and a functor $\alpha\colon I \to \mathscr{C}$ such that $A \simeq$ "colim" α.
(i) Prove that the Yoneda functor induces a fully faithful functor $\mathscr{C} \hookrightarrow \operatorname{Ind}(\mathscr{C})$, that $\operatorname{Ind}(\mathscr{C})$ admits small filtered colimits and that the functor $\operatorname{Ind}(\mathscr{C}) \hookrightarrow \mathscr{C}^\wedge$ commutes with such colimits.
(ii) Let $\mathbf{k}$ be a field and let $\mathscr{C} = \operatorname{Mod}(\mathbf{k})$. Prove that the Yoneda functor $\mathrm{h}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}^\wedge$ does not commute with colimits.

Exercise 2.11. and recall (Proposition 2.5.1) that the category $\mathscr{C}^\wedge$ admits small colimits. One denotes by "colim" the colimit in $\mathscr{C}^\wedge$. Let $\mathbf{k}$ be a field and let $\mathscr{C} = \operatorname{Mod}(\mathbf{k})$. Prove that the Yoneda functor $\mathrm{h}_{\mathscr{C}}\colon \mathscr{C} \to \mathscr{C}^\wedge$ does not commute with colimits.

Exercise 2.12. Let I be a discrete set and let $\mathscr{J}$ be the set of finite subsets of I, ordered by inclusion. We consider both I and $\mathscr{J}$ as categories. Let $\mathscr{C}$ be a category and $\alpha\colon I \to \mathscr{C}$ a functor. For $J \in \mathscr{J}$ we denote by $\alpha_J\colon J \to \mathscr{C}$ the restriction of α to J.
(i) Prove that the category $\mathscr{J}$ is filtered.
(ii) Prove the isomorphism $\operatorname*{colim}_{J\in\mathscr{J}} \operatorname*{colim}_{j\in J} \alpha_J \xrightarrow{\sim} \operatorname{colim} \alpha$.

Exercise 2.13. Let $\mathscr{C}$ be a category which admits a zero-object and kernels. Prove that a morphism $f\colon X \to Y$ is a monomorphism if and only if $\ker f \simeq 0$.

Exercise 2.14. We consider the ordered set $\mathbb{N}$ as a category. Hence, for a category $\mathscr{C}$, a functor $\alpha\colon \mathbb{N} \to \mathscr{C}$ is defined by the data of the objects $\alpha(n) \in \mathscr{C}$, $n \in \mathbb{N}$, and the morphisms $\alpha(n < n+1)\colon \alpha(n) \to \alpha(n+1)$.

(i) Consider the functor $\alpha\colon \mathbb{N} \to \mathrm{Mod}(\mathbb{Z})$ given by $\alpha(n) = \mathbb{Z}$ and $\alpha(n < n+1) = 2\cdot\colon \mathbb{Z} \to \mathbb{Z}$. Calculate $\operatorname{colim} \alpha$.
(Hint: one can represent this colimit as a a subgroup of $\mathbb{Q}$.)

(ii) Give an example of a functor $\alpha\colon \mathbb{N} \to \mathrm{Mod}(\mathbb{Z})$ in which all $\alpha(n)$ are not 0 and all morphisms $\alpha(n < n+1)$ are not 0 but $\operatorname{colim} \alpha \simeq 0$.

Exercise 2.15. Recall that **Set** denotes the category of sets in a given universe $\mathscr{U}$. Denote by $\mathbf{Set}^f$ the full subcategory of the category **Set** consisting of finite sets. Prove the equivalence $\mathrm{Ind}(\mathbf{Set}^f) \simeq \mathbf{Set}$. (See [KS06, Exa. 6.3.6].)

Exercise 2.16. Let $\mathbf{k}$ be a field and denote as usual by $\mathrm{Mod}(\mathbf{k})$ the category of $\mathbf{k}$-vector spaces (in a given universe $\mathscr{U}$). Denote by $\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$ the full subcategory consisting of finite dimensional vector spaces and set for short $\mathbf{Ik} = \mathrm{Ind}(\mathrm{Mod}(\mathbf{k}))$.

Let $\mathbb{V}$ denote an infinite dimensional vector space and denote by $\mathbb{V}^f$ the category consisting of finite dimensional vector subspaces of $\mathbb{V}$ and linear maps.

(i) Prove that the category $\mathbb{V}^f$ is small and filtered and set $\widetilde{\mathbb{V}} = \underset{W \in \mathbb{V}^f}{\text{“colim”}}\, W$.

(ii) Construct the morphism $\widetilde{\mathbb{V}} \to \mathbb{V}$ in $\mathbf{Ik}$ and prove it is a monomorphism.

(iii) Let $\mathbb{L} \in \mathrm{Mod}(\mathbf{k})$. Prove that the morphism $\mathrm{Hom}_{\mathbf{Ik}}(\mathbb{L}, \widetilde{\mathbb{V}}) \to \mathrm{Hom}_{\mathbf{Ik}}(\mathbb{L}, \mathbb{V})$ is an isomorphism if and only if $\mathbb{L} \in \mathrm{Mod}^{\mathrm{f}}(\mathbf{k})$.

(iv) Set $\mathbb{W} = \mathbb{V}/\widetilde{\mathbb{V}}$. Prove that $\mathrm{Hom}_{\mathbf{Ik}}(\mathbf{k}, \mathbb{W}) \simeq 0$ although $\mathbb{W} \neq 0$.

(v) Consider the functor $\alpha\colon \mathrm{Ind}(\mathrm{Mod}^{\mathrm{f}}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k})$ which, to $\underset{i \in I}{\text{“colim”}}\, V_i$ (I small and filtered) associates $\underset{i \in I}{\operatorname{colim}}\, V_i$. Prove that α is an equivalence of categories.

Chapter 3

Localization

[1] Consider a category $\mathscr{C}$ and a family $\mathscr{S}$ of morphisms in $\mathscr{C}$. The aim of localization is to find a new category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ which sends the morphisms belonging to $\mathscr{S}$ to isomorphisms in $\mathscr{C}_{\mathscr{S}}$, $(Q, \mathscr{C}_{\mathscr{S}})$ being "universal" for such a property.

In this chapter, we shall construct the localization of a category when $\mathscr{S}$ satisfies suitable conditions and the localization of functors. A classical reference is [GZ67].

Some proofs are skipped, references being made to [KS06].

3.1 Localization of categories

Let $\mathscr{C}$ be a category and let $\mathscr{S}$ be a family of morphisms in $\mathscr{C}$.

Definition 3.1.1. A localizaton of $\mathscr{C}$ by $\mathscr{S}$ is the data of a category $\mathscr{C}_{\mathscr{S}}$ and a functor $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ satisfying:

(a) for all $s \in \mathscr{S}$, $Q(s)$ is an isomorphism,

(b) for any functor $F\colon \mathscr{C} \to \mathscr{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathscr{S}$, there exists a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and an isomorphism $F \simeq F_{\mathscr{S}} \circ Q$,

$$\begin{tikzcd}
\mathscr{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathscr{A} \\
\mathscr{C}_{\mathscr{S}} \arrow[ur, dotted, "F_{\mathscr{S}}"'] &
\end{tikzcd}$$

(c) if G_1 and G_2 are two objects of $\mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A})$, then the natural map

$$\mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(G_1, G_2) \to \mathrm{Hom}_{\,\mathrm{Fct}(\mathscr{C},\mathscr{A})}(G_1 \circ Q, G_2 \circ Q) \tag{3.1.1}$$

is bijective.

Note that (c) means that the functor $\circ Q\colon \mathrm{Fct}(\mathscr{C}_{\mathscr{S}}, \mathscr{A}) \to \mathrm{Fct}(\mathscr{C}, \mathscr{A})$ is fully faithful. This implies that $F_{\mathscr{S}}$ in (b) is unique up to unique isomorphism.

Proposition 3.1.2. (i) *If $\mathscr{C}_{\mathscr{S}}$ exists, it is unique up to equivalence of categories.*

---

[1] The reading of this chapter may be skipped until § 6.4.

(ii) *If* $\mathscr{C}_{\mathscr{S}}$ *exists, then, denoting by* $\mathscr{S}^{\mathrm{op}}$ *the image of* $\mathscr{S}$ *in* $\mathscr{C}^{\mathrm{op}}$ *by the functor* op, $(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ *exists and there is an equivalence of categories:*

$$(\mathscr{C}_{\mathscr{S}})^{\mathrm{op}} \simeq (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}.$$

Proof. (i) is clear.
(ii) Assume $\mathscr{C}_{\mathscr{S}}$ exists. Set $(\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}} := (\mathscr{C}_{\mathscr{S}})^{\mathrm{op}}$ and define $Q^{\mathrm{op}} \colon \mathscr{C}^{\mathrm{op}} \to (\mathscr{C}^{\mathrm{op}})_{\mathscr{S}^{\mathrm{op}}}$ by $Q^{\mathrm{op}} = \mathrm{op} \circ Q \circ \mathrm{op}$. Then properties (a), (b) and (c) of Definition 3.1.1 are clearly satisfied. ☐

Definition 3.1.3. One says that $\mathscr{S}$ is a right multiplicative system if it satisfies the axioms S1-S4 below.

S1 For all $X \in \mathscr{C}$, $\mathrm{id}_X \in \mathscr{S}$.

S2 For all $f \in \mathscr{S}, g \in \mathscr{S}$, if $g \circ f$ exists then $g \circ f \in \mathscr{S}$.

S3 Given two morphisms, $f \colon X \to Y$ and $s \colon X \to X'$ with $s \in \mathscr{S}$, there exist $t \colon Y \to Y'$ and $g \colon X' \to Y'$ with $t \in \mathscr{S}$ and $g \circ s = t \circ f$. This can be visualized by the diagram:

$$\begin{tikzcd}
X' \arrow[r, dotted, "g"'] & Y' \\
X \arrow[u, "s"] \arrow[r, "f"'] & Y \arrow[u, dotted, "t"]
\end{tikzcd}$$

meaning that the dotted arrows may bwe completed, making the solid diagram commutative.

S4 Let $f, g \colon X \to Y$ be two parallel morphisms. If there exists $s \in \mathscr{S} \colon W \to X$ such that $f \circ s = g \circ s$ then there exists $t \in \mathscr{S} \colon Y \to Z$ such that $t \circ f = t \circ g$. This can be visualized by the diagram:

$$\begin{tikzcd}
W \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dotted, "t"] & Z
\end{tikzcd}$$

Notice that these axioms are quite natural if one wants to invert the elements of $\mathscr{S}$. In other words, if the element of $\mathscr{S}$ would be invertible, then these axioms would clearly be satisfied.

Remark 3.1.4. Axioms S1-S2 asserts that $\mathscr{S}$ is the family of morphisms of a subcategory $\widetilde{\mathscr{S}}$ of $\mathscr{C}$ with $\mathrm{Ob}(\widetilde{\mathscr{S}}) = \mathrm{Ob}(\mathscr{C})$.

Remark 3.1.5. One defines the notion of a left multiplicative system $\mathscr{S}$ by reversing the arrows. This means that the condition S3 is replaced by: given two morphisms, $f \colon X \to Y$ and $t \colon Y' \to Y$, with $t \in \mathscr{S}$, there exist $s \colon X' \to X$ and $g \colon X' \to Y'$ with $s \in \mathscr{S}$ and $t \circ g = f \circ s$. This can be visualized by the diagram:

$$\begin{tikzcd}
X' \arrow[r, dotted, "g"] \arrow[d, dotted, "s"'] & Y' \arrow[d, "t"'] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

meaning that the dotted arrows may bwe completed, making the solid diagram commutative.

Condition S4 is replaced by: if there exists $t \in \mathscr{S}\colon Y \to Z$ such that $t \circ f = t \circ g$ then there exists $s \in \mathscr{S}\colon W \to X$ such that $f \circ s = g \circ s$. This is visualized by the diagram

$$\begin{tikzcd} W \arrow[r, dotted, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "t"] & Z \end{tikzcd}$$

In the literature, one often calls a multiplicative system a system which is both right and left multiplicative.

Many multiplicative systems that we shall encounter satisfy a useful property that we introduce now.

Definition 3.1.6. Assume that $\mathscr{S}$ satisfies the axioms S1-S2 and let $X \in \mathscr{C}$. One defines the categories $\mathscr{S}_X$ and $\mathscr{S}^X$ as follows.

$$\begin{aligned} \mathrm{Ob}(\mathscr{S}^X) &= \{s\colon X \to X'; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}^X}((s\colon X \to X'), (s\colon X \to X'')) &= \{h\colon X' \to X''; h \circ s = s'\} \\ \mathrm{Ob}(\mathscr{S}_X) &= \{s\colon X' \to X; s \in \mathscr{S}\} \\ \mathrm{Hom}_{\mathscr{S}_X}((s\colon X' \to X), (s'\colon X'' \to X)) &= \{h\colon X' \to X''; s' \circ h = s\}. \end{aligned}$$

Note that $\mathscr{S}^X$ and $\mathscr{S}_X$ are full subcategories of $\mathscr{C}^X$ and $\mathscr{C}_X$ (see Definition 1.4.5), respectively. One shall be aware that $\mathscr{S}^X$ and $\mathscr{S}_X$ are not obtained by applying this definition to the category $\widetilde{\mathscr{S}}$ since we do not ask $h \in \mathscr{S}$.

Proposition 3.1.7. *Assume that $\mathscr{S}$ is a right (resp. left) multiplicative system. Then the category $\mathscr{S}^X$ (resp. $\mathscr{S}_X^{\mathrm{op}}$) is filtered.*

Proof. By reversing the arrows, both results are equivalent. We treat the case of $\mathscr{S}^X$.

(a) The category $\mathscr{S}^X$ is non empty since it contains id_X.

(b) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$. By S3, there exists $t\colon X' \to X'''$ and $t'\colon X'' \to X'''$ such that $t' \circ s' = t \circ s$, and $t \in \mathscr{S}$. Hence, $t \circ s \in \mathscr{S}$ by S2 and $(X \to X''')$ belongs to $\mathscr{S}^X$.

(c) Let $s\colon X \to X'$ and $s'\colon X \to X''$ belong to $\mathscr{S}$, and consider two morphisms $f, g\colon X' \to X''$, with $f \circ s = g \circ s = s'$. By S4 there exists $t\colon X'' \to W, t \in \mathscr{S}$ such that $t \circ f = t \circ g$. Hence $t \circ s'\colon X \to W$ belongs to $\mathscr{S}^X$. ☐

One defines the functors:

$$\begin{aligned} &\alpha_X\colon \mathscr{S}^X \to \mathscr{C} \qquad (s\colon X \to X') \mapsto X', \\ &\beta_X\colon \mathscr{S}_X^{\mathrm{op}} \to \mathscr{C} \qquad (s\colon X' \to X) \mapsto X'. \end{aligned}$$

We shall concentrate on right multiplicative system.

Definition 3.1.8. Let $\mathscr{S}$ be a right multiplicative system, and let $X, Y \in \mathrm{Ob}(\mathscr{C})$. We set

$$\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Y) = \operatorname*{colim}_{(Y \to Y') \in \mathscr{S}^Y} \mathrm{Hom}_{\mathscr{C}}(X, Y'). \tag{3.1.2}$$

Lemma 3.1.9. *Assume that $\mathscr{S}$ is a right multiplicative system. Let $Y \in \mathscr{C}$ and let $s\colon X \to X' \in \mathscr{S}$. Then s induces an isomorphism*

$$\operatorname{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X', Y) \xrightarrow[\circ s]{\sim} \operatorname{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Y).$$

Proof. (i) The map $\circ s$ is surjective. This follows from S3, as visualized by the diagram in which $s, t, t' \in \mathscr{S}$:

$$\begin{tikzcd}
X' \arrow[r, dotted] & Y'' & \\
X \arrow[u, "s"] \arrow[r, "f"'] & Y' \arrow[u, dotted, "t'"] & Y \arrow[l, "t"']
\end{tikzcd}$$

(ii) The map $\circ s$ is injective. This follows from S4, as visualized by the diagram in which $s, t, t' \in \mathscr{S}$:

$$\begin{tikzcd}
X \arrow[r, "s"] & X' \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y' \arrow[r, dotted, "t'"] & Y'' \\
 & & Y \arrow[u, "t"] &
\end{tikzcd}$$

$\square$

Using Lemma 3.1.9, we define the composition

$$\operatorname{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Y) \times \operatorname{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y, Z) \to \operatorname{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X, Z) \tag{3.1.3}$$

as

$$\begin{aligned}
&\operatorname*{colim}_{Y \to Y'} \operatorname{Hom}_{\mathscr{C}}(X, Y') \times \operatorname*{colim}_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}}(Y, Z') \\
&\qquad\qquad \simeq \operatorname*{colim}_{Y \to Y'} \big(\operatorname{Hom}_{\mathscr{C}}(X, Y') \times \operatorname*{colim}_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}}(Y, Z')\big) \\
&\qquad\qquad \xleftarrow{\sim} \operatorname*{colim}_{Y \to Y'} \big(\operatorname{Hom}_{\mathscr{C}}(X, Y') \times \operatorname*{colim}_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}}(Y', Z')\big) \\
&\qquad\qquad \to \operatorname*{colim}_{Y \to Y'} \operatorname*{colim}_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}}(X, Z') \\
&\qquad\qquad \simeq \operatorname*{colim}_{Z \to Z'} \operatorname{Hom}_{\mathscr{C}}(X, Z')
\end{aligned}$$

Lemma 3.1.10. *The composition* (3.1.3) *is associative.*

The verification is left to the reader.

Definition 3.1.11. (a) We denote by $\mathscr{C}^r_{\mathscr{S}}$ the category whose objects are those of $\mathscr{C}$ and morphisms are given by (3.1.2).

(b) We denote by $Q_{\mathscr{S}}\colon \mathscr{C} \to \mathscr{C}^r_{\mathscr{S}}$ the natural functor. If there is no risk of confusion, we denote this functor simply by Q.

Note that Q associated with the natural map

$$\operatorname{Hom}_{\mathscr{C}}(X, Y) \to \operatorname*{colim}_{(Y \to Y') \in \mathscr{S}^Y} \operatorname{Hom}_{\mathscr{C}}(X, Y').$$

Lemma 3.1.12. *If $s\colon X\to Y$ belongs to $\mathscr{S}$, then $Q(s)$ is invertible.*

Proof. For any $Z\in\mathscr{C}^r_{\mathscr{S}}$, the map $\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(Y,Z)\to\mathrm{Hom}_{\mathscr{C}^r_{\mathscr{S}}}(X,Z)$ is bijective by Lemma 3.1.9. □

A morphism $f\colon X\to Y$ in $\mathscr{C}^r_{\mathscr{S}}$ is thus given by an equivalence class of triplets (Y',t,f') with $t\colon Y\to Y', t\in\mathscr{S}$ and $f'\colon X\to Y'$, that is:

$$\begin{tikzcd} X \arrow[r, "f'"'] & Y' & Y \arrow[l, "t"] \end{tikzcd}$$

the equivalence relation being defined as follows: $(Y',t,f')\sim(Y'',t',f'')$ if there exists (Y''',t'',f''') $(t,t',t''\in S)$ and a commutative diagram:

$$\begin{tikzcd} & Y' \arrow[d, dotted] & \\ X \arrow[ur, "f'"] \arrow[r, dotted, "f'''"] \arrow[dr, "f''"'] & Y''' & Y \arrow[ul, "t"'] \arrow[l, dotted, "t''"'] \arrow[dl, "t'"] \\ & Y'' \arrow[u, dotted] & \end{tikzcd} \tag{3.1.4}$$

Note that the morphism (Y',t,f') in $\mathscr{C}^r_{\mathscr{S}}$ is $Q(t)^{-1}\circ Q(f')$, that is,

$$f=Q(t)^{-1}\circ Q(f'). \tag{3.1.5}$$

For two parallel arrows $f,g\colon X\rightrightarrows Y$ in $\mathscr{C}$ we have the equivalence

$$Q(f)=Q(g)\in\mathscr{C}^r_{\mathscr{S}}\Leftrightarrow \text{ there exits } s\colon Y\to Y', s\in\mathscr{S} \text{ with } s\circ f=s\circ g. \tag{3.1.6}$$

The composition of two morphisms $(Y',t,f')\colon X\to Y$ and $(Z',s,g')\colon Y\to Z$ is defined by the diagram below in which $t,s,s'\in\mathscr{S}$:

$$\begin{tikzcd} & & W & & \\ X \arrow[r, "f'"'] & Y' \arrow[ur, dotted, "h"] & Y \arrow[l, "t"] \arrow[r, "g'"'] & Z' \arrow[ul, dotted, "s'"'] & Z \arrow[l, "s"] \end{tikzcd}$$

In other words, this composition is given by $(W,s'\circ s,h\circ f')$.

Theorem 3.1.13. *Assume that $\mathscr{S}$ is a right multiplicative system. Then the category $\mathscr{C}^r_{\mathscr{S}}$ and the functor Q define a localization of $\mathscr{C}$ by $\mathscr{S}$.*

Notation 3.1.14. From now on, we shall write $\mathscr{C}_{\mathscr{S}}$ instead of $\mathscr{C}^r_{\mathscr{S}}$. This is justified by Theorem 3.1.13.

Remark 3.1.15. (i) In the above construction, we have used the property of $\mathscr{S}$ of being a right multiplicative system. If $\mathscr{S}$ is a left multiplicative system, one sets

$$\mathrm{Hom}_{\mathscr{C}^l_{\mathscr{S}}}(X,Y)=\operatorname*{colim}_{(X'\to X)\in\mathscr{S}_X}\mathrm{Hom}_{\mathscr{C}}(X',Y).$$

By Proposition 3.1.2 (i), the two constructions give equivalent categories.
(ii) If $\mathscr{S}$ is both a right and left multiplicative system,

$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{S}}}(X,Y)\ \simeq\ \operatorname*{colim}_{(X'\to X)\in\mathscr{S}_X,(Y\to Y')\in\mathscr{S}^Y}\mathrm{Hom}_{\mathscr{C}}(X',Y').$$

Remark 3.1.16. In general, $\mathscr{C}_{\mathscr{S}}$ is no more a $\mathscr{U}$-category. However, if one assumes that for any $X\in\mathscr{C}$ the category $\mathscr{S}^X$ is small (or more generally, cofinally small, which means that there exists a small category cofinal to it), then $\mathscr{C}_{\mathscr{S}}$ is a $\mathscr{U}$-category, and there is a similar result with the $\mathscr{S}_X$'s.

Saturated multiplicative systems

In this subsection, $\mathscr{C}$ is a category, $\mathscr{S}$ is a right multiplicative system and $Q\colon \mathscr{C} \to \mathscr{C}_{\mathscr{S}}$ is the localization functor.

Since the results exposed here are not used in the sequel, we leave the proofs as exercises (see also [KS06, Prop. 7.1.20]).

Lemma 3.1.17. *For a morphism $f\colon X \to Y$, $Q(f)$ is an isomorphism in $\mathscr{C}^r_{\mathscr{S}}$ if and only if there exist $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f \in \mathscr{S}$ and $h \circ g \in \mathscr{S}$.*

Definition 3.1.18. One says that $\mathscr{S}$ is saturated[2] if it satisfies

S5 for any morphisms $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f$ and $h \circ g$ belong to $\mathscr{S}$, the morphism f belongs to $\mathscr{S}$.

One deduces from Lemma 3.1.17 that if $\mathscr{S}$ is saturated, a morphism f in $\mathscr{C}$ belongs to $\mathscr{S}$ if and only if $Q(f)$ is an isomorphism.

Proposition 3.1.19. *Let $\mathscr{C}$ and $\mathscr{S}$ be as above. Let $\mathscr{T}$ be the set of morphisms $f\colon X \to Y$ in $\mathscr{C}$ such that there exist $g\colon Y \to Z$ and $h\colon Z \to W$, with $h \circ g$ and $g \circ f$ in $\mathscr{S}$. Then $\mathscr{T}$ is a right saturated multiplicative system and the natural functor $\mathscr{C}_{\mathscr{S}} \to \mathscr{C}_{\mathscr{T}}$ is an equivalence.*

The proof is left as an exeercise.

3.2 Localization of subcategories

Proposition 3.2.1. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$.*

(i) *Assume that $\mathscr{T}$ is a right multiplicative system in $\mathscr{I}$. Then the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is well-defined.*

(ii) *Assume that for every $f\colon Y \to X$, $f \in \mathscr{S}$, $Y \in \mathscr{I}$, there exist $W \in \mathscr{I}$ and $g\colon X \to W$ with $g \circ f \in \mathscr{S}$. Then $\mathscr{T}$ is a right multilplicative system and the functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful.*

Proof. (i) A morphism $X \to Y$ in $\mathscr{I}_{\mathscr{T}}$ is represented by morphisms in $\mathscr{I}$ $X \xrightarrow{f'} Y' \xleftarrow{t} Y$ with $t \in \mathscr{T}$. Since $t \in \mathscr{S}$, we get a morpism in $\mathscr{C}_{\mathscr{S}}$.
(ii) It is left to the reader to check that $\mathscr{T}$ is a right multpiplicative system. For $X \in \mathscr{I}$, $\mathscr{T}^X$ is the full subcategory of $\mathscr{S}^X$ whose objects are the morphisms $s\colon X \to Y$ with $Y \in \mathscr{I}$. By Proposition 3.1.7 and the hypothesis, the functor $\mathscr{T}^X \to \mathscr{S}^X$ is cofinal, and the result follows from Definition 3.1.8. ☐

Corollary 3.2.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Assume that for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$.*

Then $\mathscr{T}$ is a right multpiplicative system and $\mathscr{I}_{\mathscr{T}}$ is equivalent to $\mathscr{C}_{\mathscr{S}}$.

Proof. The natural functor $\mathscr{I}_{\mathscr{T}} \to \mathscr{C}_{\mathscr{S}}$ is fully faithful by Proposition 3.2.1 and is essentially surjective by the assumption. ☐

---

[2] One shall note confuse the notion of a saturated multiplicative system with that of a saturated subcategory, defined in § 1.3

3.3 Localization of functors

Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$ and $F\colon \mathscr{C} \to \mathscr{A}$ a functor. In general, F does not send morphisms in $\mathscr{S}$ to isomorphisms in $\mathscr{A}$. In other words, F does not factorize through $\mathscr{C}_{\mathscr{S}}$. It is however possible in some cases to define a localization of F as follows.

Definition 3.3.1. A right localization of F (if it exists) is a functor $F_{\mathscr{S}}\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ and a morphism of functors $\tau\colon F \to F_{\mathscr{S}} \circ Q$ such that for any functor $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ the map

$$\text{(3.3.1)} \qquad \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q)$$

is bijective. (This map is obtained as the composition $\operatorname{Hom}_{\operatorname{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G) \to \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F_{\mathscr{S}}\circ Q, G\circ Q) \xrightarrow{\tau} \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q)$.)

We shall say that F is right localizable if it admits a right localization.

One defines similarly the left localization. Since we mainly consider right localization, we shall sometimes omit the word “right” as far as there is no risk of confusion.

If $(\tau, F_{\mathscr{S}})$ exists, it is unique up to unique isomorphisms. Indeed, $F_{\mathscr{S}}$ is a representative of the functor

$$G \mapsto \operatorname{Hom}_{\operatorname{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q).$$

(This last functor is defined on the category $\operatorname{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})$ with values in **Set**.)

Proposition 3.3.2. *Let $\mathscr{C}$ be a category, $\mathscr{I}$ a full subcategory, $\mathscr{S}$ a right multiplicative system in $\mathscr{C}$, $\mathscr{T}$ the family of morphisms in $\mathscr{I}$ which belong to $\mathscr{S}$. Let $F\colon \mathscr{C} \to \mathscr{A}$ be a functor. Assume that*

(i) *for any $X \in \mathscr{C}$ there exists $s\colon X \to W$ with $W \in \mathscr{I}$ and $s \in \mathscr{S}$,*

(ii) *for any $t \in \mathscr{T}$, $F(t)$ is an isomorphism.*

Then F is right localizable.

Proof. We shall apply Corollary 3.2.2.

Denote by $\iota\colon \mathscr{I} \to \mathscr{C}$ the natural functor. By the hypothesis, the localization $F_{\mathscr{T}}$ of $F\circ\iota$ exists. Consider the diagram:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "Q_{\mathscr{S}}"] & & \mathscr{C}_{\mathscr{S}} \arrow[dd, dotted, "F_{\mathscr{S}}"] \\
\mathscr{I} \arrow[u, "\iota"] \arrow[r, "Q_{\mathscr{T}}"] \arrow[rrd, "F\circ\iota"'] & \mathscr{I}_{\mathscr{T}} \arrow[ru, "\sim", "\iota_Q"'] \arrow[rd, "F_{\mathscr{T}}"] & \\
 & & \mathscr{A}
\end{tikzcd}$$

Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $F_{\mathscr{S}} := F_{\mathscr{T}} \circ \iota_Q^{-1}$. Let us show that $F_{\mathscr{S}}$ is the localization of F. Let $G\colon \mathscr{C}_{\mathscr{S}} \to \mathscr{A}$ be a functor. We have the chain of morphisms:

$$\begin{aligned}\operatorname{Hom}_{\mathrm{Fct}(\mathscr{C},\mathscr{A})}(F, G\circ Q_{\mathscr{S}}) &\xrightarrow{\lambda} \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F\circ\iota, G\circ Q_{\mathscr{S}}\circ\iota)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I},\mathscr{A})}(F_{\mathscr{T}}\circ Q_{\mathscr{T}}, G\circ\iota_Q\circ Q_{\mathscr{T}})\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{I}_{\mathscr{T}},\mathscr{A})}(F_{\mathscr{T}}, G\circ\iota_Q)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{T}}\circ\iota_Q^{-1}, G)\\ &\simeq \operatorname{Hom}_{\mathrm{Fct}(\mathscr{C}_{\mathscr{S}},\mathscr{A})}(F_{\mathscr{S}}, G).\end{aligned}$$

We shall not prove here that λ is an isomorphism referring to [KS06, Prop. 7.3.2]. The first isomomorphism above (after λ) follows from the fact that $Q_{\mathscr{T}}$ is a localization functor (see Definition 3.1.1 (c)). The other isomorphisms are obvious. ☐

Remark 3.3.3. Let $\mathscr{C}$ (resp. $\mathscr{C}'$) be a category and $\mathscr{S}$ (resp. $\mathscr{S}'$) a right multiplicative system in $\mathscr{C}$ (resp. $\mathscr{C}'$). One checks immediately that $\mathscr{S}\times\mathscr{S}'$ is a right multiplicative system in the category $\mathscr{C}\times\mathscr{C}'$ and $(\mathscr{C}\times\mathscr{C}')_{\mathscr{S}\times\mathscr{S}'}$ is equivalent to $\mathscr{C}_{\mathscr{S}}\times\mathscr{C}'_{\mathscr{S}'}$. Since a bifunctor is a functor on the product $\mathscr{C}\times\mathscr{C}'$, we may apply the preceding results to the case of bifunctors. In the sequel, we shall write $F_{\mathscr{S}\mathscr{S}'}$ instead of $F_{\mathscr{S}\times\mathscr{S}'}$.

Exercises to Chapter 3

Exercise 3.1. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right and left multiplicative system. Prove that $\mathscr{S}$ is saturated if and only if for any $f\colon X\to Y$, $g\colon Y\to Z$, $h\colon Z\to W$, $h\circ g\in\mathscr{S}$ and $g\circ f\in\mathscr{S}$ imply $g\in\mathscr{S}$.

Exercise 3.2. Let $\mathscr{C}$ be a category with a zero object 0, $\mathscr{S}$ a right and left saturated multiplicative system.
(i) Show that $\mathscr{C}_{\mathscr{S}}$ has a zero object (still denoted by 0).
(ii) Prove that $Q(X)\simeq 0$ if and only if the zero morphism $0\colon X\to X$ belongs to $\mathscr{S}$.

Exercise 3.3. Let $\mathscr{C}$ be a category, $\mathscr{S}$ a right multiplicative system. Consider morphisms $f\colon X\to Y$ and $f'\colon X'\to Y'$ in $\mathscr{C}$ and morphisms $\alpha\colon X\to X'$ and $\beta\colon Y\to Y'$ in $\mathscr{C}_{\mathscr{S}}$, and assume that $f'\circ\alpha=\beta\circ f$ in $\mathscr{C}_{\mathscr{S}}$. Prove that there exists a commutative diagram in $\mathscr{C}$

$$\begin{tikzcd} X \arrow[r, "\alpha'"'] \arrow[rr, bend left, "\alpha"] \arrow[d, "f"'] & X_1 \arrow[d] & X' \arrow[l, "s"] \arrow[d, "f'"'] \\ Y \arrow[r, "\beta'"] \arrow[rr, bend right, "\beta"'] & Y_1 & Y' \arrow[l, "t"'] \end{tikzcd}$$

with s and t in $\mathscr{S}$, $\alpha=Q(s)^{-1}\circ Q(\alpha')$ and $\beta=Q(t)^{-1}\circ Q(\beta')$.

Exercise 3.4. Let $F\colon\mathscr{C}\to\mathscr{A}$ be a functor and assume that $\mathscr{C}$ admits finite colimits and F commutes with such colimits. Let $\mathscr{S}$ denote the set of morphisms s in $\mathscr{C}$ such that $F(s)$ is an isomorphism.
(i) Prove that $\mathscr{S}$ is a right saturated multiplicative system.
(ii) Prove that the localized functor $F_{\mathscr{S}}\colon\mathscr{C}_{\mathscr{S}}\to\mathscr{A}$ is faithful.

Chapter 4

Additive categories

Many results or constructions in the category $\mathrm{Mod}(A)$ of modules over a ring A are naturally adapted to other contexts, such as finitely generated A-modules, or graded modules over a graded ring, or sheaves of A-modules, etc. Hence, it is natural to look for a common language which avoids to repeat the same arguments. This is the language of additive and abelian categories.

In this chapter we introduce additive categories and study the category of complexes in such categories. In particular, we introduce the shifted complex, the mapping cone of a morphism, the homotopy category and the simple complex associated with a double complex, with application to bifunctors. We also briefly study the simplicial category and explain how to associate complexes to simplicial objects.

4.1 Additive categories

Definition 4.1.1. A category $\mathscr{C}$ is additive if it satisfies conditions (i)-(v) below:

(i) for any $X, Y \in \mathscr{C}$, $\quad \mathrm{Hom}_{\mathscr{C}}(X, Y) \in \mathrm{Mod}(\mathbb{Z})$,

(ii) the composition law $\circ$ is bilinear,

(iii) there exists a zero object in $\mathscr{C}$,

(iv) the category $\mathscr{C}$ admits finite coproducts,

(v) the category $\mathscr{C}$ admits finite products.

Note that $\mathrm{Hom}_{\mathscr{C}}(X, Y) \neq \varnothing$ since it is a group and for all $X \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, 0) = \mathrm{Hom}_{\mathscr{C}}(0, X) = 0$. (The morphism 0 should not be confused with the object 0.)

Notation 4.1.2. If X and Y are two objects of $\mathscr{C}$, one denotes by $X \oplus Y$ (instead of $X \coprod Y$) their coproduct, and calls it their direct sum. One denotes as usual by $X \times Y$ their product. This change of notations is motivated by the fact that if A is a ring, the forgetful functor $\mathit{for}\colon \mathrm{Mod}(A) \to \mathbf{Set}$ does not commute with coproducts.

Lemma 4.1.3. *Let $\mathscr{C}$ be a category satisfying conditions* (i)–(iii) *in* Definition 4.1.1. *Consider the condition*

(vi) *for any two objects* X *and* Y *in* $\mathscr{C}$*, there exists* $Z \in \mathscr{C}$ *and morphisms* $i_1 \colon X \to Z$*,* $i_2 \colon Y \to Z$*,* $p_1 \colon Z \to X$ *and* $p_2 \colon Z \to Y$ *satisfying*

$$p_1 \circ i_1 = \mathrm{id}_X, \quad p_1 \circ i_2 = 0 \tag{4.1.1}$$

$$p_2 \circ i_2 = \mathrm{id}_Y, \quad p_2 \circ i_1 = 0, \tag{4.1.2}$$

$$i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_Z\,. \tag{4.1.3}$$

Then the conditions (iv)*,* (v) *and* (vi) *are equivalent and the objects* $X \oplus Y$*,* $X \times Y$ *and* Z *are naturally isomorphic.*

Proof. (a) Let us assume condition (iv). The identity of X and the zero morphism $Y \to X$ define the morphism $p_1 \colon X \oplus Y \to X$ satisfying (4.1.1). We construct similarly the morphism $p_2 \colon X \oplus Y \to Y$ satisfying (4.1.2). To check (4.1.3), we use the fact that if $f \colon X \oplus Y \to X \oplus Y$ satisfies $f \circ i_1 = i_1$ and $f \circ i_2 = i_2$, then $f = \mathrm{id}_{X \oplus Y}$.
(b) Let us assume condition (vi). Let $W \in \mathscr{C}$ and consider morphisms $f \colon X \to W$ and $g \colon Y \to W$. Set $h := f \circ p_1 \oplus g \circ p_2$. Then $h \colon Z \to W$ satisfies $h \circ i_1 = f$ and $h \circ i_2 = g$ and such an h is unique. Hence $Z \simeq X \oplus Y$.
(c) We have proved that conditions (iv) and (vi) are equivalent and moreover that if they are satisfied, then $Z \simeq X \oplus Y$. Replacing $\mathscr{C}$ with $\mathscr{C}^{\mathrm{op}}$, we get that these conditions are equivalent to (v) and $Z \simeq X \times Y$. □

Example 4.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ and $\mathrm{Mod}^{\mathrm{f}}(A)$ (see Example 1.3.4) are additive categories.
(ii) **Ban**, the category of $\mathbb{C}$-Banach spaces and linear continuous maps is additive.
(iii) If $\mathscr{C}$ is additive, then $\mathscr{C}^{\mathrm{op}}$ is additive.
(iv) Let I be a small category. If $\mathscr{C}$ is additive, the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$, is additive.
(v) If $\mathscr{C}$ and $\mathscr{C}'$ are additive, then $\mathscr{C} \times \mathscr{C}'$ is additive.

Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a functor of additive categories. One says that F is additive if for $X, Y \in \mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ is a morphism of groups. We shall not prove here the following result.

Proposition 4.1.5. *Let* $F \colon \mathscr{C} \to \mathscr{C}'$ *be a functor of additive categories. Then* F *is additive if and only if it commutes with direct sum, that is, for* X *and* Y *in* $\mathscr{C}$*:*

$$\begin{aligned} F(0) &\simeq 0 \\ F(X \oplus Y) &\simeq F(X) \oplus F(Y). \end{aligned}$$

Unless otherwise specified, functors between additive categories will be assumed to be additive.
Generalization. Let $\mathbf{k}$ be a *commutative* unital ring. One defines the notion of a $\mathbf{k}$-additive category by assuming that for X and Y in $\mathscr{C}$, $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ is a $\mathbf{k}$-module and the composition is $\mathbf{k}$-bilinear.

4.2 Complexes in additive categories

Let $\mathscr{C}$ denote an additive category.

A differential object $(X^\bullet, d_X^\bullet)$ in $\mathscr{C}$ is a sequence of objects X^k and morphisms d^k $(k \in \mathbb{Z})$:

$$\cdots \to X^{k-1} \xrightarrow{d^{k-1}} X^k \xrightarrow{d^k} X^{k+1} \to \cdots . \tag{4.2.1}$$

A morphism of differential objects $f^\bullet : X^\bullet \to Y^\bullet$ is visualized by a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r] & Y^n \arrow[r, "d_Y^n"] & X^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Hence, the category $\mathrm{Diff}(\mathscr{C})$ of differential objects in $\mathscr{C}$ is nothing but the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$. In particular, it is an additive category.

Definition 4.2.1. (i) A complex in $\mathscr{C}$ is a differential object $(X^\bullet, d_X^\bullet)$ such that $d^n \circ d^{n-1} = 0$ for all $n \in \mathbb{Z}$.

(ii) One denotes by $\mathrm{C}(\mathscr{C})$ the full additive subcategory of $\mathrm{Diff}(\mathscr{C})$ consisting of complexes in $\mathscr{C}$.

From now on, we shall concentrate our study on the category $\mathrm{C}(\mathscr{C})$.

A complex is bounded (resp. bounded below, bounded above) if $X^n = 0$ for $|n| >> 0$ (resp. $n << 0$, $n >> 0$). One denotes by $\mathrm{C}^*(\mathscr{C})(* = \mathrm{b}, +, -)$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of bounded complexes (resp. bounded below, bounded above). We also use the notation $\mathrm{C}^{\mathrm{ub}}(\mathscr{C}) = \mathrm{C}(\mathscr{C})$ (ub for "unbounded"). For $a \in \mathbb{Z}$ we shall denote by $\mathrm{C}^{\geq a}(\mathscr{C})$ the full additive subcategory of $\mathrm{C}(\mathscr{C})$ consisting of objects $X^\bullet$ such that $X^j \simeq 0$ for $j < a$. One defines similarly the categories $\mathrm{C}^{\leq a}(\mathscr{C})$ and, for $a \leq b$, $\mathrm{C}^{[a,b]}(\mathscr{C})$.

One considers $\mathscr{C}$ as a full subcategory of $\mathrm{C}^{\mathrm{b}}(\mathscr{C})$ by identifying an object $X \in \mathscr{C}$ with the complex $X^\bullet$ "concentrated in degree 0":

$$X^\bullet := \cdots \to 0 \to X \to 0 \to \cdots$$

where X stands in degree 0. In other words, one identifies $\mathscr{C}$ and $\mathrm{C}^{[0,0]}(\mathscr{C})$.

Notation 4.2.2. In the definitions above of a differential object or a complex, we assumed that X^k is defined for $k \in \mathbb{Z}$. If X^k is only defined for $k \in I$, I being an interval of Z, we consider again $X^\bullet$ as a differential object or a complex by setting $X^k = 0$ for $k \notin I$.

Shift functor

Let $\mathscr{C}$ be an additive category, let $X \in \mathrm{C}(\mathscr{C})$ and let $p \in \mathbb{Z}$. One defines the shifted complex $X[p]$ by[1]:

$$\begin{aligned} (X[p])^n &= X^{n+p} \\ d_{X[p]}^n &= (-)^p d_X^{n+p} \end{aligned}$$

If $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$ one defines $f[p]\colon X[p] \to Y[p]$ by $(f[p])^n = f^{n+p}$.

The shift functor $[1]\colon X \mapsto X[1]$ is an automorphism (*i.e.* an invertible functor) of $\mathrm{C}(\mathscr{C})$.

---

[1] In these notes, we shall sometimes write $(-)^p$ instead of $(-1)^p$

Mapping cone

Definition 4.2.3. Let $f\colon X\to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. The mapping cone of f, denoted $\mathrm{Mc}(f)$, is the object of $\mathrm{C}(\mathscr{C})$ defined by:

$$\begin{aligned}\mathrm{Mc}(f)^n &= (X[1])^n\oplus Y^n\\ d^n_{\mathrm{Mc}(f)} &= \begin{pmatrix} d^n_{X[1]} & 0\\ f^{n+1} & d^n_Y\end{pmatrix}\end{aligned}$$

Of course, before to state this definition, one should check that $d^{n+1}_{\mathrm{Mc}(f)}\circ d^n_{\mathrm{Mc}(f)}=0$. Indeed:

$$\begin{pmatrix} -d^{n+2}_X & 0\\ f^{n+2} & d^{n+1}_Y\end{pmatrix}\circ\begin{pmatrix} -d^{n+1}_X & 0\\ f^{n+1} & d^{n}_Y\end{pmatrix}=0$$

Notice that although $\mathrm{Mc}(f)^n=(X[1])^n\oplus Y^n$, $\mathrm{Mc}(f)$ is not isomorphic to $X[1]\oplus Y$ in $\mathrm{C}(\mathscr{C})$ unless f is the zero morphism.

There are natural morphisms of complexes

$$(4.2.2)\qquad \alpha(f)\colon Y\to\mathrm{Mc}(f),\quad \beta(f)\colon\mathrm{Mc}(f)\to X[1].$$

and $\beta(f)\circ\alpha(f)=0$.

Example 4.2.4. Let $f\colon X\to Y$ be a morphism in $\mathscr{C}$ and let us identify $\mathscr{C}$ with a full subcategory of $\mathrm{C}(\mathscr{C})$. Then X and Y are complexes concentrated in degree 0 and f is a morphism of complexes. One checks immediately that $\mathrm{Mc}(f)$ is the complex $\cdots 0\to X\xrightarrow{f}Y\to 0\to\cdots$ where Y stands in degree 0.

If $F\colon\mathscr{C}\to\mathscr{C}'$ is an additive functor, then $F(\mathrm{Mc}(f))\simeq\mathrm{Mc}(F(f))$.

4.3 Double complexes

Let $\mathscr{C}$ be an additive category as above. A double complex $(X^{\bullet,\bullet},d_X)$ in $\mathscr{C}$ is the data of

$$\{X^{n,m},d'^{n,m}_X,d''^{n,m}_X;(n,m)\in\mathbb{Z}\times\mathbb{Z}\}$$

where $X^{n,m}\in\mathscr{C}$ and the "differentials" $d'^{n,m}_X\colon X^{n,m}\to X^{n+1,m}$, $d''^{n,m}_X : X^{n,m}\to X^{n,m+1}$ satisfy:

$$(4.3.1)\qquad d'^2_X=d''^2_X=0,\ d'\circ d''=d''\circ d'.$$

One can represent a double complex by a commutative diagram:

(4.3.2)

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & \\
{} \arrow[r] & X^{n,m} \arrow[r, "d''^{n,m}"] \arrow[d, "d'^{n,m}"'] & X^{n,m+1} \arrow[r] \arrow[d, "d'^{n,m+1}"'] & {} \\
{} \arrow[r] & X^{n+1,m} \arrow[r, "d''^{n+1,m}"'] \arrow[d] & X^{n+1,m+1} \arrow[r] \arrow[d] & {} \\
 & {} & {} &
\end{tikzcd}$$

One defines naturally the notion of a morphism of double complexes and one obtains the additive category $\mathrm{C}^2(\mathscr{C})$ of double complexes.

There is a functor $F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$ which, to a double complex X, associates the complex whose objects are the rows of X. More precisely, for $n \in \mathbb{Z}$, consider the simple complex

$$X_I^n = \{X^{n,m}, d''^{n,m}\}_{m\in\mathbb{Z}}$$

The family of morphisms $\{d'^{n,m}\}_{m\in\mathbb{Z}}$ defines a morphism $d_I^n\colon X_I^n \to X_I^{n+1}$ and one checks that $d_I^{n+1} \circ d_I^n = 0$. Therefore, $\{X_I^n, d_I^n\}_{n\in\mathbb{Z}}$ is a complex in $\mathrm{C}(\mathscr{C})$ and we have constructed the functor

$$F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C})).$$

By reversing the role of the rows and the columns, one constructs similarly the functor F_{II}. Clearly, the two functors F_I and F_{II} are isomorphisms of categories.

Assume

$$\mathscr{C} \text{ admits countable direct sums.} \tag{4.3.3}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\oplus(X)$ by setting:

$$\mathrm{tot}_\oplus(X)^p = \oplus_{m+n=p} X^{n,m}, \quad d^p_{\mathrm{tot}_\oplus(X)}|_{X^{n,m}} = d'^{n,m} + (-)^n d''^{n,m}. \tag{4.3.4}$$

This is visualized by the diagram:

$$\begin{tikzcd}
X^{n,m} \arrow[r, "(-)^n d''"] \arrow[d, "d'"'] & X^{n,m+1} \\
X^{n+1,m} &
\end{tikzcd}$$

Similarly, assume

$$\mathscr{C} \text{ admits countable products.} \tag{4.3.5}$$

One can then associate to the double complex X a simple complex $\mathrm{tot}_\pi(X)$ by setting:

$$\mathrm{tot}_\pi(X)^p = \prod_{m+n=p} X^{n,m}, \quad (d_{\mathrm{tot}_\pi(X)})^{n+m-1} = d'^{n-1,m} + (-)^n d''^{n,m-1}. \tag{4.3.6}$$

This is visualized by the diagram:

$$\begin{tikzcd}
 & X^{n-1,m} \arrow[d, "d'"'] \\
X^{n,m-1} \arrow[r, "(-)^n d''"] & X^{n,m}
\end{tikzcd}$$

One also encounters the finiteness condition:

$$\text{for all } p \in \mathbb{Z}, \quad \{(m,n) \in \mathbb{Z}\times\mathbb{Z}; X^{n,m} \neq 0, m+n=p\} \text{ is finite.} \tag{4.3.7}$$

To such an X one associates its "total complex" $\mathrm{tot}(X) = \mathrm{tot}_\oplus(X) \simeq \mathrm{tot}_\pi(X)$. In the sequel, we denote by $C_f^2(\mathscr{C})$ the full subcategory of $\mathrm{C}^2(\mathscr{C})$ consisting of objects X satisfying (4.3.7).

Proposition 4.3.1. *Assume* (4.3.3). *Then the differential object* $\{\mathrm{tot}_\oplus(X)^p, d^p_{\mathrm{tot}_\oplus(X)}\}_{p\in\mathbb{Z}}$ *is a complex* (i.e., $d^{p+1}_{\mathrm{tot}_\oplus(X)} \circ d^p_{\mathrm{tot}_\oplus(X)} = 0$) *and* $\mathrm{tot}_\oplus\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ *is a functor of additive categories.*

There is a similar result assuming (4.3.5) *or assuming that* $X \in C^2_f(\mathscr{C})$.

Proof. For short, we write simply d_{tot} or even d instead of $d_{\mathrm{tot}_\oplus(X)}$.

For $(n,m) \in \mathbb{Z}\times\mathbb{Z}$, one has

$$\begin{aligned} d\circ d|_{X^{n,m}} &= d''\circ d''|_{X^{n,m}} + d'\circ d'|_{X^{n,m}} \\ &\quad +(-)^{n+1}d''\circ d'|_{X^{n,m}} + (-)^n d'\circ d''|_{X^{n,m}} \\ &= 0. \end{aligned}$$

The fact that $\mathrm{tot}_\oplus$ is an additive functor is obvious. ☐

Example 4.3.2. Let $f^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathrm{C}(\mathscr{C})$. Consider the double complex $Z^{\bullet,\bullet}$ such that $Z^{-1,\bullet} = X^\bullet$, $Z^{0,\bullet} = Y^\bullet$, $Z^{i,\bullet} = 0$ for $i \neq -1, 0$, with differentials $f^j\colon Z^{-1,j} \to Z^{0,j}$. Then

$$\mathrm{tot}(Z^{\bullet,\bullet}) \simeq \mathrm{Mc}(f^\bullet). \tag{4.3.8}$$

Bifunctor

Let $\mathscr{C}, \mathscr{C}'$ and $\mathscr{C}''$ be additive categories and let $F\colon \mathscr{C}\times\mathscr{C}' \to \mathscr{C}''$ be an additive bifunctor (*i.e.*, $F(\bullet,\bullet)$ is additive with respect to each argument). It defines an additive bifunctor $\mathrm{C}^2(F)\colon \mathrm{C}(\mathscr{C})\times\mathrm{C}(\mathscr{C}') \to \mathrm{C}^2(\mathscr{C}'')$. In other words, if $X \in \mathrm{C}(\mathscr{C})$ and $X' \in \mathrm{C}(\mathscr{C}')$ are complexes, then $\mathrm{C}^2(F)(X,X')$ is a double complex.

Example 4.3.3. Consider the bifunctor $\bullet\otimes\bullet : \mathrm{Mod}(A^{\mathrm{op}})\times\mathrm{Mod}(A) \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall simply write $\otimes$ instead of $\mathrm{C}^2(\otimes)$. Then, for $X \in \mathrm{C}(\mathrm{Mod}(A^{\mathrm{op}}))$ and $Y \in \mathrm{C}(\mathrm{Mod}(A))$, one has

$$\begin{aligned} &(X\otimes Y)^{n,m} = X^n\otimes Y^m, \quad d'^{n,m} = d^n_X\otimes \mathrm{id}_{Y^m}, d''^{n,m} = \mathrm{id}_{X^n}\otimes d^m_Y, \\ &(\mathrm{tot}_\oplus(X,Y))^k = \bigoplus_{n+m=k} X^n\otimes Y^m, \quad d_{\mathrm{tot}(X\otimes Y)}|_{X^n\otimes Y^m} = d'^{n,m} + (-)^n d''^{n,m}. \end{aligned}$$

The complex $\mathrm{Hom}^\bullet$

Consider the bifunctor $\mathrm{Hom}_{\mathscr{C}}\colon \mathscr{C}^{\mathrm{op}}\times\mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$. In the sequel, we shall write $\mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}$ instead of $\mathrm{C}^2(\mathrm{Hom}_{\mathscr{C}})$. If X and Y are two objects of $\mathrm{C}(\mathscr{C})$, one has

$$\begin{aligned} \mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}(X,Y)^{n,m} &= \mathrm{Hom}_{\mathscr{C}}(X^{-m}, Y^n), \\ d'^{n,m} = \mathrm{Hom}_{\mathscr{C}}(X^{-m}, d^n_Y), &\quad d''^{m,n} = \mathrm{Hom}_{\mathscr{C}}((-)^m d^{-m-1}_X, Y^n). \end{aligned}$$

Note that $\mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}(X,Y)$ is a double complex in the category $\mathrm{Mod}(\mathbb{Z})$ and should not be confused with the group $\mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X,Y)$.

Let $X, Y \in \mathrm{C}(\mathscr{C})$. Using the fact that $\mathrm{Mod}(\mathbb{Z})$ admits countable products, one sets

$$\mathrm{Hom}^\bullet_{\mathscr{C}}(X,Y) = \mathrm{tot}_\pi \mathrm{Hom}^{\bullet,\bullet}_{\mathscr{C}}(X,Y), \text{ an object of } \mathrm{C}(\mathrm{Mod}(\mathbb{Z})). \tag{4.3.9}$$

Hence, $\mathrm{Hom}_{\mathscr{C}}(X,Y)^n = \prod_j \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ and $d^n\colon \mathrm{Hom}_{\mathscr{C}}(X,Y)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ is defined as follows. To $f = \{f^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j})$ one associates

$$d^n f = \{g^j\}_j \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X^j, Y^{n+j+1}), \quad g^j = d'^{n+j,-j} f^j + (-)^{j+n+1} d''^{j+n+1,-j-1} f^{j+1}.$$

In other words, the components of df in $\mathrm{Hom}_{\mathscr{C}}(X,Y)^{n+1}$ will be given by

$$(d^n f)^j = d_Y^{j+n} \circ f^j + (-)^{n+1} f^{j+1} \circ d_X^j. \tag{4.3.10}$$

Note that for $X, Y, Z \in \mathrm{C}(\mathscr{C})$, there is a natural composition map

$$\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,Y) \times \mathrm{Hom}^{\bullet}_{\mathscr{C}}(Y,Z) \xrightarrow{\circ} \mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,Z) \tag{4.3.11}$$

associated with the map

$$\begin{aligned}
&\mathrm{Hom}_{\mathscr{C}}(X,Y)^m \times \mathrm{Hom}_{\mathscr{C}}(Y,Z)^n \to \mathrm{Hom}_{\mathscr{C}}(X,Z)^{m+n},\\
&\prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Y^{i+m}) \times \prod_i \mathrm{Hom}_{\mathscr{C}}(Y^{i+m}, Z^{i+m+n}) \to \prod_i \mathrm{Hom}_{\mathscr{C}}(X^i, Z^{i+m+n}).
\end{aligned}$$

4.4 The homotopy category

Let $\mathscr{C}$ be an additive category.

Definition 4.4.1. (i) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is homotopic to zero if for all p there exists a morphism $s^p\colon X^p \to Y^{p-1}$ such that:

$$f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p.$$

Two morphisms $f, g : X \to Y$ are homotopic if $f - g$ is homotopic to zero.

(ii) An object X in $\mathrm{C}(\mathscr{C})$ is homotopic to 0 if id_X is homotopic to zero.

(iii) A morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a homotopy equivalence if there exists $g\colon Y \to X$ such that $g \circ f$ is homotopic to id_X and $f \circ g$ is homotopic to id_Y.

A morphism homotopic to zero is visualized by the diagram (which is not commutative):

$$\begin{tikzcd}
X^{p-1} \arrow[r] & X^p \arrow[r, "d_X^p"] \arrow[dl, "s^p"'] \arrow[d, "f^p"] & X^{p+1} \arrow[dl, "s^{p+1}"'] \\
Y^{p-1} \arrow[r, "d_Y^{p-1}"'] & Y^p \arrow[r] & Y^{p+1}.
\end{tikzcd}$$

Note that an additive functor sends a morphism homotopic to zero to a morphism homotopic to zero.

Example 4.4.2. (i) Let $X, Y \in \mathrm{C}(\mathscr{C})$. If both X and Y are homotopic to zero, then so is $X \oplus Y$.

(ii) Let $X \in \mathscr{C}$. Then the complex $0 \to X \xrightarrow{\mathrm{id}_X} X \to 0$ is homotopic to zero.

(iii) In particular, for $X', X'' \in \mathscr{C}$, the complex $0 \to X' \to X' \oplus X'' \to X'' \to 0$ is homotopic to zero.

Lemma 4.4.3. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two morphisms in $\mathrm{C}(\mathscr{C})$ and if f or g is homotopic to zero, then $g \circ f$ is homotopic to zero.*

Proof. Assume for example that f is homotopic to zero. In this case the proof is visualized by the diagram below.

$$\begin{tikzcd}
X^{p-1} \arrow[r] & X^p \arrow[r, "d_X^p"] \arrow[dl, "s^p"'] \arrow[d, "f^p"] & X^{p+1} \arrow[dl, "s^{p+1}"'] \\
Y^{p-1} \arrow[r] \arrow[d, "g^{p-1}"] & Y^p \arrow[r] \arrow[d, "g^p"] & Y^{p+1} \arrow[d, "g^{p+1}"] \\
Z^{p-1} \arrow[r, "d_Z^{p-1}"'] & Z^p \arrow[r] & Z^{p+1}
\end{tikzcd}$$

Indeed, the equality $f^p = s^{p+1} \circ d_X^p + d_Y^{p-1} \circ s^p$ implies

$$g^p \circ f^p = g^p \circ s^{p+1} \circ d_X^p + d_Z^{p-1} \circ g^{p-1} \circ s^p.$$

□

We shall construct a new category by deciding that a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero is isomorphic to the zero morphism. Set:

$$Ht(X, Y) = \{f\colon X \to Y; f \text{ is homotopic to } 0\}.$$

Lemma 4.4.3 allows us to state:

Definition 4.4.4. The homotopy category $\mathrm{K}(\mathscr{C})$ is defined by:

$$\begin{aligned}
\mathrm{Ob}(\mathrm{K}(\mathscr{C})) &= \mathrm{Ob}(\mathrm{C}(\mathscr{C})) \\
\mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X, Y) &= \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X, Y)/Ht(X, Y).
\end{aligned}$$

In other words, a morphism homotopic to zero in $\mathrm{C}(\mathscr{C})$ becomes the zero morphism in $\mathrm{K}(\mathscr{C})$ and a homotopy equivalence becomes an isomorphism.

One defines similarly $\mathrm{K}^*(\mathscr{C})$, $(* = \mathrm{ub}, \mathrm{b}, +, -)$. They are clearly additive categories endowed with an automorphism, the shift functor $[1]\colon X \mapsto X[1]$.

Recall (4.3.9).

Proposition 4.4.5. *Let $\mathscr{C}$ be an additive category and let $X, Y \in \mathrm{C}(\mathscr{C})$. There are isomorphisms:*

$$\begin{aligned}
Z^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X, Y)) &:= \ker d^0 \simeq \mathrm{Hom}_{\mathrm{C}(\mathscr{C})}(X, Y), \\
B^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X, Y)) &:= \mathrm{Im}\, d^{-1} \simeq Ht(X, Y), \\
H^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X, Y)) &:= \ker d^0 / \mathrm{Im}\, d^{-1} \simeq \mathrm{Hom}_{\mathrm{K}(\mathscr{C})}(X, Y).
\end{aligned}$$

Proof. (i) Let us calculate $Z^0(\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,Y))$. By (4.3.10), the component of $d^0\{f^j\}_j$ in $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j+1})$ will be zero if and only if $d^j_Y \circ f^j = f^{j+1} \circ d^j_X$, that is, if the family $\{f^j\}_j$ defines a morphism of complexes.
(ii) Let us calculate $B^0(\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,Y))$. An element $f^j \in \mathrm{Hom}_{\mathscr{C}}(X^j, Y^j)$ will be in the image of d^{-1} if it is in the sum of the image of $\mathrm{Hom}_{\mathscr{C}}(X^j, Y^{j-1})$ by d^{j-1}_Y and the image of $\mathrm{Hom}_{\mathscr{C}}(X^{j+1}, Y^j)$ by d^j_X. Hence, if it can be written as $f^j = d^{j-1}_Y \circ s^j + s^{j+1} \circ d^j_X$.
(iii) The third isomorphism follows. ☐

Remark 4.4.6. The preceding constructions could be developed in the general setting of DG-categories. Roughly speaking, a DG-category is an additive category in which the morphisms are no more additive groups but are complexes of such groups.

The category $\mathrm{C}(\mathscr{C})$ endowed for each $X, Y \in \mathrm{C}(\mathscr{C})$ with the complex $\mathrm{Hom}^{\bullet}_{\mathscr{C}}(X,Y))$ and the composition being given by (4.3.11) is an example of such a DG-category. More more details on this subject, see for example [Kel06, Yek20].

We shall come back to the category $\mathrm{K}(\mathscr{C})$ in § 6.3.

4.5 Simplicial constructions

We shall define the simplicial category and use it to construct complexes and homotopies in additive categories.

Definition 4.5.1. (a) The simplicial category, denoted by $\mathbf{\Delta}$, is the category whose objects are the finite totally ordered sets and the morphisms are the order-preserving maps.

(b) We denote by $\mathbf{\Delta}_{inj}$ the subcategory of $\mathbf{\Delta}$ such that $\mathrm{Ob}(\mathbf{\Delta}_{inj}) = \mathrm{Ob}(\mathbf{\Delta})$, the morphisms being the injective order-preserving maps.

For integers n, m denote by $[n, m]$ the totally ordered set $\{k \in \mathbb{Z}; n \leq k \leq m\}$.

Proposition 4.5.2. (i) *the natural functor* $\mathbf{\Delta} \to \mathbf{Set}^f$ *is faithful,*

(ii) *the full subcategory of* $\mathbf{\Delta}$ *consisting of objects* $\{[0,n]\}_{n \geq -1}$ *is equivalent to* $\mathbf{\Delta}$,

(iii) $\mathbf{\Delta}$ *admits an initial object, namely* $\varnothing$, *and a terminal object, namely* $\{0\}$.

The proof is obvious.
Let us denote by

$$d^n_i \colon [0,n] \to [0,n+1] \qquad (0 \leq i \leq n+1)$$

the injective order-preserving map which does not take the value i. In other words

$$d^n_i(k) = \begin{cases} k & \text{for } k < i, \\ k+1 & \text{for } k \geq i. \end{cases}$$

---

[1] § 4.5 may be skipped.

One checks immediately that

$$d_j^{n+1} \circ d_i^n = d_i^{n+1} \circ d_{j-1}^n \text{ for } 0 \le i < j \le n+2. \tag{4.5.1}$$

Indeed, both morphisms are the unique injective order-preserving map which does not take the values i and j.

The category $\mathbf{\Delta}_{inj}$ is visualized by

$$\begin{tikzcd} \varnothing \arrow[r, "d_0^{-1}" description] & {[0]} \arrow[r, shift left=1.5, "d_0^0" description] \arrow[r, shift right=1.5, "d_1^0" description] & {[0,1]} \arrow[r, shift left=3, "d_0^1" description] \arrow[r, "d_1^1" description] \arrow[r, shift right=3, "d_2^1" description] & {[0,1,2]} \arrow[r, dotted, shift left=4.5] \arrow[r, dotted, shift left=1.5] \arrow[r, dotted, shift right=1.5] \arrow[r, dotted, shift right=4.5] & {} \end{tikzcd} \tag{4.5.2}$$

Let $\mathscr{C}$ be an additive category and $F\colon \mathbf{\Delta}_{inj} \to \mathscr{C}$ a functor. We set for $n \in \mathbb{Z}$:

$$F^n = \begin{cases} F([0,n]) & \text{for } n \ge -1, \\ 0 & \text{otherwise,} \end{cases}$$

$$d_F^n\colon F^n \to F^{n+1}, \quad d_F^n = \sum_{i=0}^{n+1} (-)^i F(d_i^n).$$

Consider the differential object

$$F^\bullet := \cdots \to 0 \to F^{-1} \xrightarrow{d_F^{-1}} F^0 \xrightarrow{d_F^0} F^1 \to \cdots \to F^n \xrightarrow{d_F^n} \cdots. \tag{4.5.3}$$

Theorem 4.5.3. (i) *The differential object $F^\bullet$ is a complex.*

(ii) *Assume that there exist morphisms $s_F^n\colon F^n \to F^{n-1}$ $(n \ge 0)$satisfying:*

$$\begin{cases} s_F^{n+1} \circ F(d_0^n) = \mathrm{id}_{F^n} & \textit{for } n \ge -1, \\ s_F^{n+1} \circ F(d_{i+1}^n) = F(d_i^{n-1}) \circ s_F^n & \textit{for } i > 0, n \ge 0. \end{cases}$$

Then $F^\bullet$ is homotopic to zero.

Proof. (i) By (4.5.1), we have

$$\begin{aligned} d_F^{n+1} \circ d_F^n &= \sum_{j=0}^{n+2} \sum_{i=0}^{n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j} F(d_j^{n+1} \circ d_i^n) + \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) \\ &= 0\,. \end{aligned}$$

Here, we have used

$$\begin{aligned} \sum_{0 \le i < j \le n+2} (-)^{i+j} F(d_i^{n+1} \circ d_{j-1}^n) &= \sum_{0 \le i < j \le n+1} (-)^{i+j+1} F(d_i^{n+1} \circ d_j^n) \\ &= \sum_{0 \le j \le i \le n+1} (-)^{i+j+1} F(d_j^{n+1} \circ d_i^n). \end{aligned}$$

(ii) We have

$$\begin{aligned}
&s_F^{n+1}\circ d_F^n + d_F^{n-1}\circ s^n\\
&\quad=\sum_{i=0}^{n+1}(-1)^i s_F^{n+1}\circ F(d_i^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad= s_F^{n+1}\circ F(d_0^n)+\sum_{i=0}^{n}(-1)^{i+1}s_F^{n+1}\circ F(d_{i+1}^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad= \mathrm{id}_{F^n}+\sum_{i=0}^{n}(-1)^{i+1}F(d_i^{n-1}\circ s_F^n)+\sum_{i=0}^{n}(-1)^i F(d_i^{n-1}\circ s_F^n)\\
&\quad= \mathrm{id}_{F^n}.
\end{aligned}$$

□

Exercises to Chapter 4

Exercise 4.1. Let $\mathscr{C}$ be an additive category and let $X \in \mathrm{C}(\mathscr{C})$ with differential d_X. Define the morphism $\delta_X\colon X \to X[1]$ by setting $\delta_X^n = (-1)^n d_X^n$. Prove that δ_X is a morphism in $\mathrm{C}(\mathscr{C})$ and is homotopic to zero.

Exercise 4.2. (see [KS06, Exe. 11.4].) Let $\mathscr{C}$ be an additive category, $f, g\colon X \rightrightarrows Y$ two morphisms in $\mathrm{C}(\mathscr{C})$. Prove that f and g are homotopic if and only if there exists a commutative diagram in $\mathrm{C}(\mathscr{C})$

$$\begin{tikzcd}
Y \arrow[r, "\alpha(f)"'] \arrow[d, equal] & \mathrm{Mc}(f) \arrow[r, "\beta(f)"'] \arrow[d, "u"] & X[1] \arrow[d, equal]\\
Y \arrow[r, "\alpha(g)"'] & \mathrm{Mc}(g) \arrow[r, "\beta(g)"'] & X[1].
\end{tikzcd}$$

In such a case, prove that u is an isomorphism in $\mathrm{C}(\mathscr{C})$.

Exercise 4.3. (see [KS06, Exe. 11.6].) Let $\mathscr{C}$ be an additive category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$.
Prove that the following conditions are equivalent:

(a) f is homotopic to zero,

(b) f factors through $\alpha(\mathrm{id}_X)\colon X \to \mathrm{Mc}(\mathrm{id}_X)$,

(c) f factors through $\beta(\mathrm{id}_Y)[-1]\colon \mathrm{Mc}(\mathrm{id}_Y)[-1] \to Y$,

(d) f decomposes as $X \to Z \to Y$ with Z a complex homotopic to zero.

Exercise 4.4. (See [KS06, § 10.1].) A category with translation $(\mathscr{A}, T)$ is a category $\mathscr{A}$ together with an equivalence $T\colon \mathscr{A} \to \mathscr{A}$. A differential object (X, d_X) in a category with translation $(\mathscr{A}, T)$ is an object $X \in \mathscr{A}$ together with a morphism $d_X\colon X \to T(X)$. A morphism $f\colon (X, d_X) \to (Y, d_Y)$ of differential objects is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "d_X"] \arrow[d, "f"] & TX \arrow[d, "T(f)"]\\
Y \arrow[r, "d_Y"] & TY.
\end{tikzcd}$$

One denotes by $\mathscr{A}_d$ the category consisting of differential objects and morphisms of such objects. If $\mathscr{A}$ is additive, one says that a differential object (X, d_X) in $(\mathscr{A}, T)$ is a complex if the composition $X \xrightarrow{d_X} T(X) \xrightarrow{T(d_X)} T^2(X)$ is zero. One denotes by $\mathscr{A}_c$ the full subcategory of $\mathscr{A}_d$ consisting of complexes.
(i) Let $\mathscr{C}$ be a category. Denote by $\mathbb{Z}_d$ the set $\mathbb{Z}$ considered as a discrete category and still denote by $\mathbb{Z}$ the ordered set $(\mathbb{Z}, \leq)$ considered as a category. Prove that $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$ is a category with translation.
(ii) Show that the category $\mathrm{Fct}(\mathbb{Z}, \mathscr{C})$ may be identified to the category of differential objects in $\mathrm{Fct}(\mathbb{Z}_d, \mathscr{C})$.
(iii) Let $\mathscr{C}$ be an additive category. Show that the notions of differential objects and complexes given above coincide with those in Definition 4.2.1 when choosing $\mathscr{A} = \mathrm{C}(\mathscr{C})$ and $T = [1]$.

Exercise 4.5. Consider the catgeory $\mathbf{\Delta}$ and for $n > 0$, denote by

$$s_i^n \colon [0, n] \to [0, n-1] \qquad (0 \leq i \leq n-1)$$

the surjective order-preserving map which takes the same value at i and $i+1$. In other words

$$s_i^n(k) = \begin{cases} k & \text{for } k \leq i, \\ k-1 & \text{for } k > i. \end{cases}$$

Checks the relations:

$$\begin{cases} s_j^n \circ s_i^{n+1} = s_{i-1}^n \circ s_j^{n+1} & \text{for } 0 \leq j < i \leq n, \\ s_j^{n+1} \circ d_i^n = d_i^{n-1} \circ s_{j-1}^n & \text{for } 0 \leq i < j \leq n, \\ s_j^{n+1} \circ d_i^n = \mathrm{id}_{[0,n]} & \text{for } 0 \leq i \leq n+1, i = j, j+1, \\ s_j^{n+1} \circ d_i^n = d_{i-1}^{n-1} \circ s_j^n & \text{for } 1 \leq j+1 < i \leq n+1. \end{cases}$$

Chapter 5

Abelian categories

The toy model of abelian categories is the category $\mathrm{Mod}(A)$ of modules over a ring A and for sake of simplicity, we shall argue most of the time as if we were working in a full abelian subcategory of a category $\mathrm{Mod}(A)$. This is not restrictive in view of a famous theorem of Fred&Mitchell [Mit60, Fre64].

We introduce injective and projective objects and state without proof the famous Grothendieck theorem which asserts that what is now called a Grothendieck category admits enough injectives.

We explain the notions of exact sequences and right or left exact functors, we give some basic lemmas such as "the five lemma" and "the snake lemma", we construct the long exact sequence associated with an exact sequence of complexes and we also study double complexes.

Finally, we study with some details Koszul complexes and show how they naturally appear in Algebra and Analysis.

Some references: see [CE56, Gro57] for historical references and [Wei94, KS06] for an exposition. Here we shall often follow this last reference.

5.1 Abelian categories

Let $\mathscr{C}$ be an additive category which admits kernels and cokernels (recall Definition 2.2.1). Equivalently, $\mathscr{C}$ admits finite limits and colimits.

Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$. We have already defined the image and co-image of f in Definition 2.4.4. Denote by $h\colon \ker f \to X$ and $k\colon Y \to \mathrm{Coker}\, f$ the natural morphisms.

Lemma 5.1.1. *One has the isomorphisms*

$$\mathrm{Coim}\, f \simeq \mathrm{Coker}\, h,\ \mathrm{Im}\, f \simeq \ker k.$$

Proof. Of course, it is enough to prove the first isomorphism. For $Z \in \mathscr{C}$, one has (see Diagram 2.2.6)

$$\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) = \{u\colon X \to Z; u \circ p_1 = u \circ p_2\},$$

where $p_1, p_2\colon X \times_Y X \to X$ are the two projections. Since $X \times_Y X$ is the kernel of $(f \circ p_1, f \circ p_2)\colon X \times X \rightrightarrows Y$, one also have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(\mathrm{Coim}\, f, Z) = \{u\colon X \to Z; u \circ v_1 = u \circ v_2 \text{ for any } W \text{ and } (v_1, v_2)\colon W \rightrightarrows X \\ \text{such that } f \circ v_1 = f \circ v_2.\}\end{aligned}$$

Equivalently,

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) = \{u\colon X \to Z; u \circ v = 0 \text{ for any } W \text{ and } v\colon W \to X \\ \text{such that} f \circ v = 0.\}\end{aligned}$$

Since such a v factorizes uniquely through h, we get

$$\begin{aligned}\mathrm{Hom}_{\mathscr{C}}(\operatorname{Coim} f, Z) &= \{u\colon X \to Z; u \circ h = 0\} \\ &\simeq \mathrm{Hom}_{\mathscr{C}}(\operatorname{Coker} h, Z).\end{aligned}$$

Since this isomorphism is functorial in Z (this point being left to the reader), we get the result by the Yoneda lemma. □

Consider the diagram:

$$\begin{tikzcd}
\ker f \arrow[r, "h"] & X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "k"] & \operatorname{Coker} f \\
& \operatorname{Coim} f \arrow[r, dotted, "u"] \arrow[ur, dotted, "\tilde{f}"] & \operatorname{Im} f \arrow[u] &
\end{tikzcd}$$

Since $f \circ h = 0$, f factors uniquely through $\operatorname{Coim} f$, which defines $\tilde{f}$ (see Diagram 2.2.6) and thus $k \circ f$ factors through $k \circ \tilde{f}$. Since $k \circ f = k \circ \tilde{f} \circ s = 0$ and s is an epimorphism, we get that $k \circ \tilde{f} = 0$. Hence $\tilde{f}$ factors through $\ker k = \operatorname{Im} f$, which defines u (see Diagram 2.2.5). We have thus constructed a canonical morphism:

$$\operatorname{Coim} f \xrightarrow{u} \operatorname{Im} f. \tag{5.1.1}$$

Examples 5.1.2. (i) For a ring A and a morphism f in $\mathrm{Mod}(A)$, (5.1.1) is an isomorphism.

(ii) The category **Ban** admits kernels and cokernels. If $f\colon X \to Y$ is a morphism of Banach spaces, define $\ker f = f^{-1}(0)$ and $\operatorname{Coker} f = Y/\overline{\operatorname{Im} f}$ where $\overline{\operatorname{Im} f}$ denotes the closure of the space $\operatorname{Im} f$. It is well-known that there exist continuous linear maps $f\colon X \to Y$ which are injective, with dense and non closed image. For such an f, $\ker f = \operatorname{Coker} f = 0$ although f is not an isomorphism. Thus $\operatorname{Coim} f \simeq X$ and $\operatorname{Im} f \simeq Y$. Hence, the morphism (5.1.1) is not an isomorphism.

(iii) Let A be a ring, I an ideal which is not finitely generated and let $M = A/I$. Then the natural morphism $A \to M$ in $\mathrm{Mod}^{\mathrm{f}}(A)$ has no kernel.

Definition 5.1.3. Let $\mathscr{C}$ be an additive category. One says that $\mathscr{C}$ is abelian if:

(i) any $f\colon X \to Y$ admits a kernel and a cokernel,

(ii) for any morphism f in $\mathscr{C}$, the natural morphism $\operatorname{Coim} f \to \operatorname{Im} f$ is an isomorphism.

Examples 5.1.4. (i) If A is a ring, $\mathrm{Mod}(A)$ is an abelian category. If A is noetherian, then $\mathrm{Mod}^{\mathrm{f}}(A)$ is abelian.

(ii) The category **Ban** admits kernels and cokernels but is not abelian. (See Examples 5.1.2 (ii).)

(iii) If $\mathscr{C}$ is abelian, then $\mathscr{C}^{\mathrm{op}}$ is abelian.

Proposition 5.1.5. *Let I be category and let $\mathscr{C}$ be an abelian category. Then the category $\mathrm{Fct}(I, \mathscr{C})$ of functors from I to $\mathscr{C}$ is abelian.*

Proof. (i) Let $F, G\colon I \to \mathscr{C}$ be two functors and $\varphi\colon F \to G$ a morphism of functors. Let us define a new functor H as follows. For $i \in I$, set $H(i) = \ker(F(i) \to G(i))$. Let $s\colon i \to j$ be a morphism in I. In order to define the morphism $H(s)\colon H(i) \to H(j)$, consider the diagram

$$\begin{tikzcd}
H(i) \arrow[r, "h_i"] \arrow[d, dotted, "H(s)"'] & F(i) \arrow[r, "\varphi(i)"] \arrow[d, "F(s)"'] & G(i) \arrow[d, "G(s)"] \\
H(j) \arrow[r, "h_j"] & F(j) \arrow[r, "\varphi(i)"] & G(j).
\end{tikzcd}$$

Since $\varphi(j) \circ F(s) \circ h_i = 0$, the morphism $F(s) \circ h_i$ factorizes uniquely through $H(j)$. This gives $H(s)$. One checks immediately that for a morphism $t\colon j \to k$ in I, one has $H(t) \circ H(s) = H(t \circ s)$. Therefore H is a functor and one also easily cheks that H is a kernel of the morphism of functors φ.
(ii) One defines similarly the functor $\mathrm{Coim}\,\varphi$. Since, for each $i \in I$, the natural morphism $\mathrm{Coim}\,\varphi(i) \to \mathrm{Im}\,\varphi(i)$ is an isomorphism, one deduces that the natural morphism of functors $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. ☐

Corollary 5.1.6. *If $\mathscr{C}$ is abelian, then the categories of complexes $\mathrm{C}^*(\mathscr{C})$ $(* = \mathrm{ub}, \mathrm{b}, +, -)$ are abelian.*

Proof. It follows from Proposition 5.1.5 that the category $\mathrm{Diff}(\mathscr{C})$ of differential objects of $\mathscr{C}$ is abelian. One checks immediately that if $f^\bullet\colon X^\bullet \to Y^\bullet$ is a morphism of complexes, its kernel in the category $\mathrm{Diff}(\mathscr{C})$ is a complex and is a kernel in the category $\mathrm{C}(\mathscr{C})$, and similarly with cokernels. ☐

For example, if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, the complex Z defined by $Z^n = \ker(f^n\colon X^n \to Y^n)$, with differential induced by those of X, will be a kernel for f, and similarly for $\mathrm{Coker}\, f$.

Note the following results.

- An abelian category admits finite limits and finite colimits. (Indeed, an abelian category admits an initial object, a terminal object, finite products and finite coproducts and kernels and cokernels.)

- In an abelian category, a morphism f is a monomorphism (resp. an epimorphism) if and only if $\ker f \simeq 0$ (resp. $\mathrm{Coker}\, f \simeq 0$) (see Exercise 2.13). Moreover, a morphism $f\colon X \to Y$ is an isomorphism as soon as $\ker f \simeq 0$ and $\mathrm{Coker}\, f \simeq 0$. Indeed, in such a case, $X \xrightarrow{\sim} \mathrm{Coim}\, f$ and $\mathrm{Im}\, f \xrightarrow{\sim} Y$.

Unless otherwise specified, we assume until the end of this chapter that $\mathscr{C}$ is abelian.

Consider a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ (hence, $g \circ f = 0$). It defines a morphism $\mathrm{Coim}\, f \to \ker g$, hence, $\mathscr{C}$ being abelian, a morphism $\mathrm{Im}\, f \to \ker g$.

Definition 5.1.7. (i) One says that a complex $X' \xrightarrow{f} X \xrightarrow{g} X''$ is exact if $\mathrm{Im}\, f \xrightarrow{\sim} \ker g$.

(ii) More generally, a sequence of morphisms $X^p \xrightarrow{d^p} \cdots \to X^n$ with $d^{i+1} \circ d^i = 0$ for all $i \in [p, n-1]$ is exact if $\operatorname{Im} d^i \xrightarrow{\sim} \ker d^{i+1}$ for all $i \in [p, n-1]$.

(iii) A short exact sequence is an exact sequence $0 \to X' \to X \to X'' \to 0$

Any morphism $f \colon X \to Y$ may be decomposed into short exact sequences:

$$\begin{aligned} &0 \to \ker f \to X \to \operatorname{Coim} f \to 0, \\ &0 \to \operatorname{Im} f \to Y \to \operatorname{Coker} f \to 0, \end{aligned}$$

with $\operatorname{Coim} f \simeq \operatorname{Im} f$.

Proposition 5.1.8. *Let*

$$0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0 \tag{5.1.2}$$

be a short exact sequence in $\mathscr{C}$. *Then the conditions* (a) *to* (e) *are equivalent.*

(a) *there exists* $h \colon X'' \to X$ *such that* $g \circ h = \mathrm{id}_{X''}$.

(b) *there exists* $k \colon X \to X'$ *such that* $k \circ f = \mathrm{id}_{X'}$.

(c) *there exists* $\varphi = (k, g)$ *and* $\psi = \begin{pmatrix} f \\ h \end{pmatrix}$ *such that* $X \xrightarrow{\varphi} X' \oplus X''$ *and* $X' \oplus X'' \xrightarrow{\psi} X$ *are isomorphisms inverse to each other.*

(d) *The complex* (5.1.2) *is homotopic to* 0.

(e) *The complex* (5.1.2) *is isomorphic to the complex* $0 \to X' \to X' \oplus X'' \to X'' \to 0$.

Proof. (a) ⇒ (c). Since $g = g \circ h \circ g$, we get $g \circ (\mathrm{id}_X - h \circ g) = 0$, which implies that $\mathrm{id}_X - h \circ g$ factors through $\ker g$, that is, through X'. Hence, there exists $k \colon X \to X'$ such that $\mathrm{id}_X - h \circ g = f \circ k$.
(b) ⇒ (c) follows by reversing the arrows.
(c) ⇒ (a). Since $g \circ f = 0$, we find $g = g \circ h \circ g$, that is $(g \circ h - \mathrm{id}_{X''}) \circ g = 0$. Since g is an epimorphism, this implies $g \circ h - \mathrm{id}_{X''} = 0$.
(c) ⇒ (b) follows by reversing the arrows.
(d) By definition, the complex (5.1.2) is homotopic to zero if and only if there exists a diagram

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & X \arrow[r, "g"] \arrow[d, "\mathrm{id}"] \arrow[dl, "k"'] & X'' \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, "h"'] & 0 \\
0 \arrow[r] & X' \arrow[r, "f"'] & X \arrow[r, "g"'] & X'' \arrow[r] & 0
\end{tikzcd}$$

such that $\mathrm{id}_{X'} = k \circ f$, $\mathrm{id}_{X''} = g \circ h$ and $\mathrm{id}_X = h \circ g + f \circ k$.
(e) is obvious by (c). □

Definition 5.1.9. In the above situation, one says that the exact sequence splits.

Note that an additive functor of abelian categories sends split exact sequences to split exact sequences.

If $\mathscr{C} = \mathrm{Mod}(\mathbf{k})$ and $\mathbf{k}$ is a field, then all exact sequences split, but this is not the case in general.

Example 5.1.10. The exact sequence of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

does not split.

5.2 Exact functors

We recall here Definition 2.6.8 in the particular case of additive categories.

Definition 5.2.1. Let $F \colon \mathscr{C} \to \mathscr{C}'$ be an additive functor of abelian categories. One says that

(i) F is left exact if it commutes with finite limits,

(ii) F is right exact if it commutes with finite colimits,

(iii) F is exact if it is both left and right exact.

Lemma 5.2.2. *Consider an additive functor $F \colon \mathscr{C} \to \mathscr{C}'$.*

(a) *The conditions below are equivalent:*

  (i) *F is left exact,*
  (ii) *F commutes with kernels, that is, for any morphism $f \colon X \to Y$, $F(\ker(f)) \xrightarrow{\sim} \ker(F(f))$,*
  (iii) *for any exact sequence $0 \to X' \to X \to X''$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*
  (iv) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$.*

(b) *The conditions below are equivalent:*

  (i) *F is exact,*
  (ii) *for any exact sequence $X' \to X \to X''$ in $\mathscr{C}$, the sequence $F(X') \to F(X) \to F(X'')$ is exact in $\mathscr{C}'$,*
  (iii) *for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact in $\mathscr{C}'$.*

There is a similar result to (a) for right exact functors.

Proof. Since F is additive, it commutes with terminal objects and products of two objects. Hence, by Proposition 2.3.9, F is left exact if and only if it commutes with kernels.
The proof of the other assertions are left as an exercise. $\square$

Proposition 5.2.3. (i) *The functor* $\mathrm{Hom}_{\mathscr{C}} \colon \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \to \mathrm{Mod}(\mathbb{Z})$ *is left exact with respect to each of its arguments.*

(ii) *If a functor* $F \colon \mathscr{C} \to \mathscr{C}'$ *admits a left* (*resp. right*) *adjoint then* F *is left* (*resp. right*) *exact.*

(iii) *Let* I *be a small category. If* $\mathscr{C}$ *admits limits indexed by* I*, then the functor* $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathscr{C}) \to \mathscr{C}$ *is left exact. Similarly, if* $\mathscr{C}$ *admits colimits indexed by* I*, then the functor* $\mathrm{colim} : \mathrm{Fct}(I, \mathscr{C}) \to \mathscr{C}$ *is right exact.*

(iv) *Let* A *be a ring and let* I *be a small set. The two functors* $\prod_{i\in I}$ *and* $\bigoplus_{i\in I}$ *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *are exact.*

(v) *Let* A *be a ring and let* I *be a small filtered category. The functor* colim *from* $\mathrm{Fct}(I, \mathrm{Mod}(A))$ *to* $\mathrm{Mod}(A)$ *is exact.*

Proof. (i) follows from (2.3.2) and (2.3.3).
(ii) Apply Proposition 2.5.5.
(iii) Apply Proposition 2.5.1.
(iv) is left as an exercise (see Exercise 5.1).
(v) follows from Corollary 2.6.7. ☐

Example 5.2.4. Let A be a ring and let N be a right A-module. Since the functor $N \otimes_A \bullet$ admits a right adjoint, it is right exact. Let us show that the functors $\mathrm{Hom}_A(\bullet, \bullet)$ and $N \otimes_A \bullet$ are not exact in general. In the sequel, we choose $A = \mathbf{k}[x]$, with $\mathbf{k}$ a field, and we consider the exact sequence of A-modules:

$$0 \to A \xrightarrow{\cdot x} A \to A/Ax \to 0, \tag{5.2.1}$$

where $\cdot x$ means multiplication by x.
(i) Apply the functor $\mathrm{Hom}_A(\bullet, A)$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to A \xrightarrow{x\cdot} A \to 0$$

which is not exact since $x\cdot$ is not surjective. On the other hand, since $x\cdot$ is injective and $\mathrm{Hom}_A(\bullet, A)$ is left exact, we find that $\mathrm{Hom}_A(A/Ax, A) = 0$.
(ii) Apply $\mathrm{Hom}_A(A/Ax, \bullet)$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A) \to \mathrm{Hom}_A(A/Ax, A/Ax) \to 0.$$

Since $\mathrm{Hom}_A(A/Ax, A) = 0$ and $\mathrm{Hom}_A(A/Ax, A/Ax) \neq 0$, this sequence is not exact.
(iii) Apply $\bullet \otimes_A A/Ax$ to the exact sequence (5.2.1). We get the sequence:

$$0 \to A/Ax \xrightarrow{x\cdot} A/Ax \to A/xA \otimes_A A/Ax \to 0.$$

Multiplication by x is 0 on A/Ax. Hence this sequence is the same as:

$$0 \to A/Ax \xrightarrow{0} A/Ax \to A/Ax \otimes_A A/Ax \to 0$$

which shows that $A/Ax \otimes_A A/Ax \simeq A/Ax$ and moreover that this sequence is not exact.
(iv) Notice that the functor $\mathrm{Hom}_A(\bullet, A)$ being additive, it sends split exact sequences to split exact sequences. This shows that (5.2.1) does not split.

Example 5.2.5. We shall show that the functor $\lim : \mathrm{Fct}(I^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k})$ is not right exact in general, even if $\mathbf{k}$ is a field.

Consider as above the $\mathbf{k}$-algebra $A := \mathbf{k}[x]$ over a field $\mathbf{k}$. Denote by $I = A \cdot x$ the ideal generated by x. Notice that $A/I^{n+1} \simeq \mathbf{k}[x]^{\leq n}$, where $\mathbf{k}[x]^{\leq n}$ denotes the $\mathbf{k}$-vector space consisting of polynomials of degree $\leq n$. For $p \leq n$ denote by $v_{pn} \colon A/I^n \twoheadrightarrow A/I^p$ the natural epimorphisms. They define a projective system of A-modules. One checks easily that

$$\lim_n A/I^n \simeq \mathbf{k}[[x]],$$

the ring of formal series with coefficients in $\mathbf{k}$. On the other hand, for $p \leq n$ the monomorphisms $I^n \rightarrowtail I^p$ define a projective system of A-modules and one has

$$\lim_n I^n \simeq 0.$$

Now consider the projective system of exact sequences of A-modules

$$0 \to I^n \to A \to A/I^n \to 0.$$

By taking the limit of these exact sequences one gets the sequence $0 \to 0 \to \mathbf{k}[x] \to \mathbf{k}[[x]] \to 0$ which is no more exact, neither in the category $\mathrm{Mod}(A)$ nor in the category $\mathrm{Mod}(\mathbf{k})$.

The Mittag-Leffler condition

Let us give a criterion in order that the limit of an exact sequence remains exact in the category $\mathrm{Mod}(A)$. This is a particular case of the so-called "Mittag-Leffler" condition (see [EGA3]).

Proposition 5.2.6. *Let A be a ring and let $0 \to \{M'_n\} \xrightarrow{f_n} \{M_n\} \xrightarrow{g_n} \{M''_n\} \to 0$ be an exact sequence of projective systems of A-modules indexed by $\mathbb{N}$. Assume that for each n, the map $M'_{n+1} \to M'_n$ is surjective. Then the sequence*

$$0 \to \lim_n M'_n \xrightarrow{f} \lim_n M_n \xrightarrow{g} \lim_n M''_n \to 0$$

is exact.

Proof. Let us denote by v_p the morphisms $M_p \to M_{p-1}$ which define the projective system $\{M_p\}$, and similarly for v'_p, v''_p. Let $\{x''_p\}_p \in \lim_n M''_n$. Hence $x''_p \in M''_p$, and $v''_p(x''_p) = x''_{p-1}$.

We shall first show that $v_n \colon g_n^{-1}(x''_n) \to g_{n-1}^{-1}(x''_{n-1})$ is surjective. Let $x_{n-1} \in g_{n-1}^{-1}(x''_{n-1})$. Take $x_n \in g_n^{-1}(x''_n)$. Then $g_{n-1}(v_n(x_n) - x_{n-1})) = 0$. Hence $v_n(x_n) - x_{n-1} = f_{n-1}(x'_{n-1})$. By the hypothesis $f_{n-1}(x'_{n-1}) = f_{n-1}(v'_n(x'_n))$ for some x'_n and thus $v_n(x_n - f_n(x'_n)) = x_{n-1}$.

Then we can choose $x_n \in g_n^{-1}(x''_n)$ inductively such that $v_n(x_n) = x_{n-1}$. $\square$

5.3 Injective and projective objects

Definition 5.3.1. Let $\mathscr{C}$ be an abelian category.

(i) An object I of $\mathscr{C}$ is injective if the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ is exact.

(ii) One says that $\mathscr{C}$ has enough injectives if for any $X \in \mathscr{C}$ there exists a monomorphism $X \rightarrowtail I$ with I injective.

(iii) An object P is projective in $\mathscr{C}$ if it is injective in $\mathscr{C}^{\mathrm{op}}$, *i.e.,* if the functor $\mathrm{Hom}_{\mathscr{C}}(P, \,\bullet\,)$ is exact.

(iv) One says that $\mathscr{C}$ has enough projectives if for any $X \in \mathscr{C}$ there exists an epimorphism $P \twoheadrightarrow X$ with P projective.

Proposition 5.3.2. *The object $I \in \mathscr{C}$ is injective if and only if, for any $X, Y \in \mathscr{C}$ and any diagram in which the row is exact:*

$$\begin{tikzcd}
0 \arrow[r] & X' \arrow[r, "f"] \arrow[d, "k"'] & X \arrow[ld, dotted, "h"'] \\
 & I &
\end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

Proof. (i) Assume that I is injective and let X'' denote the cokernel of the morphism $X' \to X$. Applying the functor $\mathrm{Hom}_{\mathscr{C}}(\,\bullet\,, I)$ to the sequence $0 \to X' \to X \to X''$, one gets the exact sequence:

$$\mathrm{Hom}_{\mathscr{C}}(X'', I) \to \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0.$$

Thus there exists $h \colon X \to I$ such that $h \circ f = k$.

(ii) Conversely, consider an exact sequence $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$. Then the sequence $0 \to \mathrm{Hom}_{\mathscr{C}}(X'', I) \xrightarrow{\circ h} \mathrm{Hom}_{\mathscr{C}}(X, I) \xrightarrow{\circ f} \mathrm{Hom}_{\mathscr{C}}(X', I) \to 0$ is exact by the hypothesis.

To conclude, apply Lemma 5.2.2. ☐

By reversing the arrows, we get that P is projective if and only if for any diagram in which the row is exact:

$$\begin{tikzcd}
 & P \arrow[d, "k"] \arrow[ld, dotted, "h"'] & \\
X \arrow[r, "f"] & X'' \arrow[r] & 0
\end{tikzcd}$$

the dotted arrow may be completed, making the solid diagram commutative.

Lemma 5.3.3. *Let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$, and assume that X' is injective. Then the sequence splits.*

Proof. Applying the preceding result with $k = \mathrm{id}_{X'}$, we find $h \colon X \to X'$ such that $k \circ f = \mathrm{id}_{X'}$. Then apply Proposition 5.1.8. ☐

It follows that if $F\colon \mathscr{C} \to \mathscr{C}'$ is an additive functor of abelian categories, and the hypotheses of the lemma are satisfied, then the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ splits and in particular is exact.

Lemma 5.3.4. *Let X', X'' belong to $\mathscr{C}$. Then $X' \oplus X''$ is injective if and only if X' and X'' are injective.*

Proof. It is enough to remark that for two additive functors of abelian categories F and G, the functor $F \oplus G\colon X \mapsto F(X) \oplus G(X)$ is exact if and only if the functors F and G are exact. $\square$

Applying Lemmas 5.3.3 and 5.3.4, we get:

Proposition 5.3.5. *Let $0 \to X' \to X \to X'' \to 0$ be an exact sequence in $\mathscr{C}$ and assume X' and X are injective. Then X'' is injective.*

Example 5.3.6. (i) Let A be a ring. An A-module M is free if it is isomorphic to a direct sum of copies of A, that is, $M \simeq A^{\oplus I}$ for some small set I. It follows from (2.1.4) and Proposition 5.2.3 (iv) that free modules are projective.

Let $M \in \mathrm{Mod}(A)$. For $m \in M$, denote by A_m a copy of A and denote by $1_m \in A_m$ the unit. Define the linear map

$$\psi\colon \bigoplus_{m\in M} A_m \to M$$

by setting $\psi(1_m) = m$ and extending by linearity. This map is clearly surjective. Since the left A-module $\bigoplus_{m\in M} A_m$ is free, it is projective. This shows that the category $\mathrm{Mod}(A)$ has enough projectives.

More generally, if there exists an A-module N such that $M \oplus N$ is free then M is projective (see Exercise 5.3).

One can prove that $\mathrm{Mod}(A)$ has enough injectives (see Exercise 5.4).

(ii) If $\mathbf{k}$ is a field, then any object of $\mathrm{Mod}(\mathbf{k})$ is both injective and projective.

(iii) Let A be a $\mathbf{k}$-algebra and let $M \in \mathrm{Mod}(A^{\mathrm{op}})$. One says that M is flat if the functor $M \otimes_A \bullet\colon \mathrm{Mod}(A) \to \mathrm{Mod}(\mathbf{k})$ is exact. Clearly, projective modules are flat.

Although Proposition 5.3.7 below is a particular case of Theorem 7.2.2, we include it for pedagogical reasons.

For a category $\mathscr{C}$, denote by $\mathscr{I}_{\mathscr{C}}$ the full additive subcategory of injective objects.

Proposition 5.3.7. *Let $\mathscr{C}$ be an abelian category which admits enough injectives. Then, for any $X \in \mathscr{C}$, there exists an exact sequence*

$$0 \to X \to I_0^0 \to \cdots \to I_X^n \to \cdots \tag{5.3.1}$$

with $I_X^n \in \mathscr{I}_{\mathscr{C}}$ for all $n \geq 0$.

Proof. We proceed by induction. Assume to have constructed:

$$0 \to X \to I_X^0 \to \cdots \to I_X^n.$$

For $n = 0$ this is the hypothesis. Set $B^n = \mathrm{Coker}(I_X^{n-1} \to I_X^n)$ (with $I_X^{-1} = X$). Then $I_X^{n-1} \to I_X^n \to B^n \to 0$ is exact. Embed B^n in an injective object: $0 \to B^n \to I_X^{n+1}$. Then $I_X^{n-1} \to I_X^n \to I_X^{n+1}$ is exact, and the induction proceeds. $\square$

The sequence

$$I_X^\bullet := 0 \to I_X^0 \to \cdots \to I_X^n \to \cdots \tag{5.3.2}$$

is called an injective resolution resolution of X.

Note that, identifying X and $I_X^\bullet$ to objects of $\mathrm{C}^+(\mathscr{C})$,

$$X \to I_X^\bullet \text{ is a qis.} \tag{5.3.3}$$

Of course, there is a similar result for projective resolutions. If for any $X \in \mathscr{C}$ there is an exact sequence $Y \to X \to 0$ with Y projective, then one can construct a projective-resolution of X, that is, a qis $P_X^\bullet \to X$, where the P_X^n's are projective.

5.4 Generators and Grothendieck categories

In this section it is essential to fix a universe $\mathscr{U}$. Hence, a category means a $\mathscr{U}$-category and small means $\mathscr{U}$-small.

Definition 5.4.1. Let $\mathscr{C}$ be a category. A system of generators in $\mathscr{C}$ is a family of objects $\{G_i\}_{i\in I}$ of $\mathscr{C}$ such that I is small and a morphism $f\colon X \to Y$ in $\mathscr{C}$ is an isomorphism as soon as $\mathrm{Hom}_{\mathscr{C}}(G_i, X) \to \mathrm{Hom}_{\mathscr{C}}(G_i, Y)$ is an isomorphism for all $i \in I$.

If the family contains a single element, say G, one says that G is a generator.

If $\{G_i\}_{i\in I}$ is a system of generators, then the functor $\prod_{i\in I} \mathrm{Hom}_{\mathscr{C}}(G_i, \bullet)\colon \mathscr{C} \to$ **Set** is conservative. If $\mathscr{C}$ is additive, these two conditions are equivalent [1]. Moreover, if $\mathscr{C}$ is additive, admits small coproducts and a system of generators as above, then it admits a generator, namely the object $\bigoplus_{i\in I} G_i$.

Lemma 5.4.2. *Let $\mathscr{C}$ be an abelian category which admits small coproducts and a generator G.*

(a) *The functor $\mathrm{Hom}_{\mathscr{C}}(G, \bullet)$ is faithful.*

(b) *For any $X \in \mathscr{C}$, there exist a small set I and an epimorphism $G^{\oplus I} \twoheadrightarrow X$.*

Proof. In this proof, we write $\mathrm{Hom}\,(Y, Z)$ instead of $\mathrm{Hom}_{\mathscr{C}}(Y, Z)$.

(a) The functor $\mathrm{Hom}_{\mathscr{C}}(G, \bullet)$ is left exact and conservative by the hypothesis. Then use Exercise 5.13.

(b) There is a natural isomorphism (see Exercise 5.12):

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Hom}\,(G, X), \mathrm{Hom}\,(G, X)) \simeq \mathrm{Hom}\,(G^{\oplus \mathrm{Hom}\,(G,X)}, X).$$

The identity morphism on the left-hand side defines the natural morphism $G^{\oplus \mathrm{Hom}\,(G,X)} \to X$. This morphism defines the morphism

$$\mathrm{Hom}\,(G, G^{\oplus \mathrm{Hom}\,(G,X)}) \to \mathrm{Hom}\,(G, X).$$

This last morphism being obviously surjective, the result follows from Exercise 5.14. □

---

[1] There was a mistake in [KS06, Def. 5.2.1], see the Errata on the webpage of the author.

Definition 5.4.3. A Grothendieck category is an abelian category which admits small limits and small colimits, a generator and such that filtered small colimits are exact.

We shall not give the proof of the important Grothendieck's theorem below, referring to [KS06, Th. 9.6.2]. See [Gro57] for the original proof.

Theorem 5.4.4. *Let $\mathscr{C}$ be an abelian Grothendieck category. Then $\mathscr{C}$ admits enough injectives.*

5.5 Complexes in abelian categories

One still denotes by $\mathscr{C}$ an abelian category.

Solving linear equations

The aim of this subsection is to illustrate and motivate the constructions which will appear further. In this subsection, we work in the category $\mathrm{Mod}(A)$ for a $\mathbf{k}$-algebra A. Recall that the category $\mathrm{Mod}(A)$ admits enough projectives.

Suppose one is interested in studying a system of linear equations

$$\sum_{j=1}^{N_0} p_{ij}u_j = v_i, \quad (i = 1, \dots, N_1) \tag{5.5.1}$$

where the p_{ij}'s belong to the ring A and u_j, v_i belong to some left A-module S. Using matrix notations, one can write equations (5.5.1) as

$$P_0 u = v \tag{5.5.2}$$

where P_0 is the matrix (p_{ij}) with N_1 rows and N_0 columns, defining the A-linear map $P_0\cdot : S^{N_0} \to S^{N_1}$. Now consider the right A-linear map

$$\cdot P_0 : A^{N_1} \to A^{N_0}, \tag{5.5.3}$$

where $\cdot P_0$ operates on the right and the elements of A^{N_0} and A^{N_1} are written as rows. Let $(e_1, \dots, e_{N_0})$ and $(f_1, \dots, f_{N_1})$ denote the canonical basis of A^{N_0} and A^{N_1}, respectively. One gets:

$$f_i \cdot P_0 = \sum_{j=1}^{N_0} p_{ij}e_j, \quad (i = 1, \dots, N_1). \tag{5.5.4}$$

Hence $\mathrm{Im}\, P_0$ is generated by the elements $\sum_{j=1}^{N_0} p_{ij}e_j$ for $i = 1, \dots, N_1$. Denote by M the quotient module $A^{N_0}/A^{N_1} \cdot P_0$ and by $\psi : A^{N_0} \to M$ the natural A-linear map. Let $(u_1, \dots, u_{N_0})$ denote the images by ψ of $(e_1, \dots, e_{N_0})$. Then M is a left A-module with generators $(u_1, \dots, u_{N_0})$ and relations $\sum_{j=1}^{N_0} p_{ij}u_j = 0$ for $i = 1, \dots, N_1$. By construction, we have an exact sequence of left A-modules:

$$A^{N_1} \xrightarrow{\cdot P_0} A^{N_0} \xrightarrow{\psi} M \to 0. \tag{5.5.5}$$

Applying the left exact functor $\operatorname{Hom}_A(\,\bullet\,,S)$ to this sequence, we find the exact sequence of **k**-modules:

$$0\to\operatorname{Hom}_A(M,S)\to S^{N_0}\xrightarrow{P_0\cdot}S^{N_1}\tag{5.5.6}$$

(where $P_0\cdot$ operates on the left). Hence, the **k**-module of solutions of the homogeneous equation associated to (5.5.1) is described by $\operatorname{Hom}_A(M,S)$.

Assume now that A is left Noetherian, that is, any submodule of a free A-module of finite rank is of finite type. In this case, arguing as in the proof of Proposition 5.3.7 below, we construct an exact sequence

$$\cdots\to A^{N_2}\xrightarrow{\cdot P_1}A^{N_1}\xrightarrow{\cdot P_0}A^{N_0}\xrightarrow{\psi}M\to 0.$$

In other words, we have a projective resolution $L^\bullet\to M$ of M by finite free left A-modules:

$$L^\bullet:\ \cdots\to L^n\to L^{n-1}\to\cdots\to L^0\to 0.$$

Applying the left exact functor $\operatorname{Hom}_A(\,\bullet\,,S)$ to $L^\bullet$, we find the complex of **k**-modules:

$$0\to S^{N_0}\xrightarrow{P_0\cdot}S^{N_1}\xrightarrow{P_1\cdot}S^{N_2}\to\cdots\,.\tag{5.5.7}$$

Then

$$\begin{cases}H^0(\operatorname{Hom}_A(L^\bullet,S))\simeq\ker P_0,\\ H^1(\operatorname{Hom}_A(L^\bullet,S))\simeq\ker(P_1)/\operatorname{Im}(P_0).\end{cases}$$

Hence, a necessary condition to solve the equation $P_0u=v$ is that $P_1v=0$ and this necessary condition is sufficient if $H^1(\operatorname{Hom}_A(L^\bullet,S))\simeq 0$. As we shall see in § 7.3, the cohomology groups $H^j(\operatorname{Hom}_A(L^\bullet,S))$ do not depend, up to isomorphisms, of the choice of the projective resolution $L^\bullet$ of M and are denoted $\operatorname{Ext}^j_A(M,S)$.

Cohomology

Recall that the categories $\mathrm{C}^*(\mathscr{C})$ are abelian for $*=\mathrm{ub},+,-,\mathrm{b}$.

Let $X\in\mathrm{C}(\mathscr{C})$. One defines the following objects of $\mathscr{C}$:

$$\begin{aligned}Z^n(X)&:=\ker d^n_X\\ B^n(X)&:=\operatorname{Im}d^{n-1}_X\\ H^n(X)&:=Z^n(X)/B^n(X)\quad(:=\operatorname{Coker}(B^n(X)\to Z^n(X)))\end{aligned}$$

One calls $H^n(X)$ the n-th cohomology object of X. If $f\colon X\to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, then it induces morphisms $Z^n(X)\to Z^n(Y)$ and $B^n(X)\to B^n(Y)$, thus a morphism $H^n(f)\colon H^n(X)\to H^n(Y)$. Clearly, $H^n(X\oplus Y)\simeq H^n(X)\oplus H^n(Y)$. Hence we have obtained an additive functor:

$$H^n(\,\bullet\,):\mathrm{C}(\mathscr{C})\to\mathscr{C}.$$

Notice that $H^n(X)=H^0(X[n])$.

There are exact sequences

$$X^{n-1}\xrightarrow{d^{n-1}}\ker d^n_X\to H^n(X)\to 0,$$
$$0\to H^n(X)\to\operatorname{Coker}d^{n-1}_X\xrightarrow{d^n}X^{n+1}.$$

The next result is easily checked.

Lemma 5.5.1. *For $n \in \mathbb{Z}$, the sequences below are exact:*

$$0 \to H^n(X) \to \mathrm{Coker}(d_X^{n-1}) \xrightarrow{d_X^n} \ker d_X^{n+1} \to H^{n+1}(X) \to 0. \tag{5.5.8}$$

One defines the truncation functors:

$$\begin{aligned} \tau^{\leq n}, \widetilde{\tau}^{\leq n} &\colon C(\mathscr{C}) \to C^-(\mathscr{C}) \\ \tau^{\geq n}, \widetilde{\tau}^{\geq n} &\colon C(\mathscr{C}) \to C^+(\mathscr{C}) \end{aligned} \tag{5.5.9}$$

as follows. Let $X := \cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots$. One sets:

$$\begin{aligned} \tau^{\leq n}(X) &:= \cdots \to X^{n-2} \to X^{n-1} \to \ker d_X^n \to 0 \to \cdots \\ \widetilde{\tau}^{\leq n}(X) &:= \cdots \to X^{n-1} \to X^n \to \operatorname{Im} d_X^n \to 0 \to \cdots \\ \tau^{\geq n}(X) &:= \cdots \to 0 \to \operatorname{Coker} d_X^{n-1} \to X^{n+1} \to X^{n+2} \to \cdots \\ \widetilde{\tau}^{\geq n}(X) &:= \cdots \to 0 \to \operatorname{Im} d_X^{n-1} \to X^n \to X^{n+1} \to \cdots \end{aligned}$$

There is a chain of morphisms in $\mathrm{C}(\mathscr{C})$:

$$\tau^{\leq n} X \to \widetilde{\tau}^{\leq n} X \to X \to \widetilde{\tau}^{\geq n} X \to \tau^{\geq n} X,$$

and there are exact sequences in $\mathrm{C}(\mathscr{C})$:

$$\begin{cases} 0 \to \widetilde{\tau}^{\leq n-1} X \to \tau^{\leq n} X \to H^n(X)[-n] \to 0, \\ 0 \to H^n(X)[-n] \to \tau^{\geq n} X \to \widetilde{\tau}^{\geq n+1} X \to 0, \\ 0 \to \tau^{\leq n} X \to X \to \widetilde{\tau}^{\geq n+1} X \to 0, \\ 0 \to \widetilde{\tau}^{\leq n-1} X \to X \to \tau^{\geq n} X \to 0. \end{cases} \tag{5.5.10}$$

We have the isomorphisms

$$\begin{aligned} H^j(\tau^{\leq n} X) &\xrightarrow{\sim} H^j(\widetilde{\tau}^{\leq n} X) \simeq \begin{cases} H^j(X) & j \leq n, \\ 0 & j > n. \end{cases} \\ H^j(\widetilde{\tau}^{\geq n} X) &\xrightarrow{\sim} H^j(\tau^{\geq n} X) \simeq \begin{cases} H^j(X) & j \geq n, \\ 0 & j < n. \end{cases} \end{aligned} \tag{5.5.11}$$

The verification is straightforward.

Lemma 5.5.2. *Let $\mathscr{C}$ be an abelian category and let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$ homotopic to zero. Then $H^n(f)\colon H^n(X) \to H^n(Y)$ is the 0 morphism.*

Proof. Let $f^n = s^{n+1} \circ d_X^n + d_Y^{n-1} \circ s^n$. Then $d_X^n = 0$ on $\ker d_X^n$ and $d_Y^{n-1} \circ s^n = 0$ on $\ker d_Y^n / \operatorname{Im} d_Y^{n-1}$. Hence $H^n(f)\colon \ker d_X^n / \operatorname{Im} d_X^{n-1} \to \ker d_Y^n / \operatorname{Im} d_Y^{n-1}$ is the zero morphism. □

In view of Lemma 5.5.2, the functor $H^0\colon \mathrm{C}(\mathscr{C}) \to \mathscr{C}$ extends as a functor

$$H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}.$$

One shall be aware that the additive category $\mathrm{K}(\mathscr{C})$ is not abelian in general.

Definition 5.5.3. One says that a morphism $f\colon X \to Y$ in $\mathrm{C}(\mathscr{C})$ is a quasi-isomorphism (a qis, for short) if $H^k(f)$ is an isomorphism for all $k \in \mathbb{Z}$. In such a case, one says that X and Y are quasi-isomorphic. In particular, $X \in \mathrm{C}(\mathscr{C})$ is qis to 0 if and only if the complex X is exact.

Remark 5.5.4. By Lemma 5.5.2, a complex homotopic to 0 is qis to 0, but the converse is false. In particular, the property for a complex of being homotopic to 0 is preserved when applying an additive functor, contrarily to the property of being qis to 0.

Remark 5.5.5. Consider a bounded complex $X^\bullet$ and denote by $Y^\bullet$ the complex given by $Y^j = H^j(X^\bullet), d_Y^j \equiv 0$. One has:

$$Y^\bullet = \bigoplus_i H^i(X^\bullet)[-i]. \tag{5.5.12}$$

The complexes $X^\bullet$ and $Y^\bullet$ have the same cohomology objects. In other words, $H^j(Y^\bullet) \simeq H^j(X^\bullet)$. However, in general these isomorphisms are neither induced by a morphism from $X^\bullet \to Y^\bullet$, nor by a morphism from $Y^\bullet \to X^\bullet$, and the two complexes $X^\bullet$ and $Y^\bullet$ are not quasi-isomorphic.

Long exact sequence

Lemma 5.5.6. (The "five lemma".) *Consider a commutative diagram:*

$$\begin{tikzcd}
X^0 \arrow[r, "\alpha_0"] \arrow[d, "f^0"'] & X^1 \arrow[r, "\alpha_1"] \arrow[d, "f^1"'] & X^2 \arrow[r, "\alpha_2"] \arrow[d, "f^2"'] & X^3 \arrow[d, "f^3"'] \\
Y^0 \arrow[r, "\beta_0"'] & Y^1 \arrow[r, "\beta_1"'] & Y^2 \arrow[r, "\beta_2"'] & Y^3
\end{tikzcd}$$

and assume that the rows are exact.

(i) *If f^0 is an epimorphism and f^1, f^3 are monomorphisms, then f^2 is a monomorphism.*

(ii) *If f^3 is a monomorphism and f^0, f^2 are epimorphisms, then f^1 is an epimorphism.*

As already mentioned in the introduction of this Chapter, there is a theorem of Fred and Mitchell [Mit60, Fre64] which asserts that we may assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A, what we will do here. Hence we may choose elements in the objects of $\mathscr{C}$.

Proof. (i) Let $x_2 \in X_2$ and assume that $f^2(x_2) = 0$. Then $f^3 \circ \alpha_2(x_2) = 0$ and f^3 being a monomorphism, this implies $\alpha_2(x_2) = 0$. Since the first row is exact, there exists $x_1 \in X_1$ such that $\alpha_1(x_1) = x_2$. Set $y_1 = f^1(x_1)$. Since $\beta_1 \circ f^1(x_1) = 0$ and the second row is exact, there exists $y_0 \in Y^0$ such that $\beta_0(y_0) = f^1(x_1)$. Since f^0 is an epimorphism, there exists $x_0 \in X^0$ such that $y_0 = f^0(x_0)$. Since $f^1 \circ \alpha_0(x_0) = f^1(x_1)$ and f^1 is a monomorphism, $\alpha_0(x_0) = x_1$. Therefore, $x_2 = \alpha_1(x_1) = 0$.
(ii) is nothing but (i) in $\mathscr{C}^{\mathrm{op}}$. ☐

Lemma 5.5.7. (The snake lemma.) *Consider the commutative diagram in $\mathscr{C}$ below with exact rows:*

$$\begin{tikzcd}
 & X' \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[r, "g"] \arrow[d, "\beta"'] & X'' \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & Y' \arrow[r, "f'"] & Y \arrow[r, "g'"] & Y'' &
\end{tikzcd}$$

Then there exists a morphism $\delta\colon \ker\gamma \to \operatorname{Coker}\alpha$ *giving rise to an exact sequence:*

$$\ker\alpha \to \ker\beta \to \ker\gamma \xrightarrow{\delta} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma. \tag{5.5.13}$$

Proof. Here again, we shall assume that $\mathscr{C}$ is a full abelian subcategory of $\mathrm{Mod}(A)$ for some ring A.

(i) Let us first prove that the sequence $\ker\alpha \to \ker\beta \to \ker\gamma$ is exact. Let $x \in \ker\beta$ with $g(x) = 0$. Using the fact that the first row is exact, there exists $x' \in X'$ with $f(x') = x$. Then $f' \circ \alpha(x') = \beta \circ f(x') = 0$. Since f' is a monomorphism, $\alpha(x') = 0$ and $x' \in \ker\alpha$.

(ii) The sequence $\operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma$ is exact. If one works in the abstract setting of abelian categories, this follows from (i) by reversing the arrows. Otherwise, if one wishes to remain in the setting of A-modules, one can adapt the proof of (i)[2].

(iii) Let us construct the map δ making the sequence exact. Let $x'' \in \ker\gamma$ and choose $x \in X$ with $g(x) = x''$. Set $y = \beta(y)$. Since $g'(y) = 0$, there exists $y' \in Y'$ with $f'(y') = y$. One defines $\delta(x'')$ as the image of y' in $\operatorname{Coker}\alpha$, that is, in $Y'/\operatorname{Im}\alpha$.

The reader will check that the map δ is well-defined (i.e., the construction does not depend on the various choices) and that the sequence (5.5.13) is exact. ☐

One shall be aware that the morphism δ is not unique. Replacing δ with $-\delta$ does not change the conclusion.

Theorem 5.5.8. *Let* $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ *be an exact sequence in* $\mathrm{C}(\mathscr{C})$.

(i) *For each* $k \in \mathbb{Z}$, *the sequence* $H^k(X') \to H^k(X) \to H^k(X'')$ *is exact.*

(ii) *For each* $k \in \mathbb{Z}$, *there exists* $\delta^k : H^k(X'') \to H^{k+1}(X')$ *making the long sequence*

$$\cdots \to H^k(X) \to H^k(X'') \xrightarrow{\delta^k} H^{k+1}(X') \to H^{k+1}(X) \to \cdots \tag{5.5.14}$$

exact. Moreover, one can construct δ^k *functorial with respect to short exact sequences of* $\mathrm{C}(\mathscr{C})$.

---

[2] The reader shall be aware that the opposite of an abelian category is still abelian, but the category $\mathrm{Mod}(A)$ is not equivalent to the opposite category $\mathrm{Mod}(A)^{\mathrm{op}}$.

Proof. Consider the commutative diagrams:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & H^k(X') \arrow[d] & H^k(X) \arrow[d] & H^k(X'') \arrow[d] & \\
 & \operatorname{Coker} d_{X'}^{k-1} \arrow[r, "f"'] \arrow[d, "d_{X'}^k"'] & \operatorname{Coker} d_X^{k-1} \arrow[r, "g"'] \arrow[d, "d_X^k"'] & \operatorname{Coker} d_{X''}^{k-1} \arrow[r] \arrow[d, "d_{X''}^k"'] & 0 \\
0 \arrow[r] & \ker d_{X'}^{k+1} \arrow[r, "f"'] \arrow[d] & \ker d_X^{k+1} \arrow[r, "g"'] \arrow[d] & \ker d_{X''}^{k+1} \arrow[d] & \\
 & H^{k+1}(X') \arrow[d] & H^{k+1}(X) \arrow[d] & H^{k+1}(X'') \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The columns are exact by Lemma 5.5.1 and the rows are exact by the hypothesis. Hence, the result follows from Lemma 5.5.7. ☐

Corollary 5.5.9. *Consider a morphism* $f\colon X \to Y$ *in* $\mathrm{C}(\mathscr{C})$ *and recall that* $\mathrm{Mc}(f)$ *denotes the mapping cone of* f. *There is a long exact sequence:*

$$\cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^k(X) \xrightarrow{f} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots . \tag{5.5.15}$$

Proof. Using (4.2.2), we get a complex:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0. \tag{5.5.16}$$

Clearly, this complex is exact. Indeed, in degree n, it gives the split exact sequence $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$. Applying Theorem 5.5.8, we find a long exact sequence

$$\cdots \to H^{k-1}(\mathrm{Mc}(f)) \to H^{k-1}(X[1]) \xrightarrow{\delta^{k-1}} H^k(Y) \to H^k(\mathrm{Mc}(f)) \to \cdots . \tag{5.5.17}$$

It remains to check that, up to a sign, the morphism $\delta^{k-1}\colon H^k(X) \to H^k(Y)$ is $H^k(f)$. We shall not give the proof here. ☐

One shall be aware that although the exact sequences $0 \to Y^n \to Y^n \oplus X^{n+1} \to X^{n+1} \to 0$ split, the exact sequence of complexes (5.5.16) does not split in general.

5.6 Double complexes in abelian categories

In this subsection we shall illustrate the fact that the use of truncation functors is an alternative to that of spectral sequences (and is much easier). We follow [KS06, § 12.5].

Let $\mathscr{C}$ denote an abelian category.

Recall that, for a double complex $X = X^{\bullet,\bullet}$, the finiteness condition (4.3.7) says that for all $p \in \mathbb{Z}$, the set $\{(m,n) \in \mathbb{Z} \times \mathbb{Z}; m + n = p \text{ such that } X^{n,m} \neq 0\}$ is finite. From now on,

(5.6.1) We assume that X satisfies (4.3.7).

Note that

(5.6.2) The functor $\mathrm{tot}\colon \mathrm{C}^2_f(\mathscr{C}) \to \mathrm{C}(\mathscr{C})$ is exact.

In Section 4.3, we have constructed the functors $F_I\colon \mathrm{C}^2(\mathscr{C}) \to \mathrm{C}(\mathrm{C}(\mathscr{C}))$. Since now $\mathscr{C}$ is abelian, we can consider the truncation functors $\tau_I^{\leq n}$, $\tilde{\tau}_I^{\leq n}$, $\tau_I^{\geq n}$, etc. For example, one defines $\tau_I^{\leq n} := F_I^{-1} \circ \tau^{\leq n} \circ F_I$, that is, setting $X_I = F_I(X)$:

$$\tau_I^{\leq n}(X) = \cdots \to X_I^{n-1} \xrightarrow{d_I^{n-1}} X_I^n \to \ker d_I^n \to 0.$$

For $n \in \mathbb{Z}$, we also introduce the simple complex

$$H_I^n(X) = H^n(F_I(X))$$

and the double complex

$$H_I(X) = \cdots \to X_I^{n-1} \xrightarrow{0} X_I^n \to \cdots .$$

Of course, the same constructions hold with F_{II} instead of F_I.

It follows from (5.5.11) and (5.6.2) that

(5.6.3) the natural morphism $\mathrm{tot}(\tau_I^{\leq n}(X)) \to \mathrm{tot}(\tilde{\tau}_I^{\leq n}(X))$ is a qis for all n.

Since $H_I^n(X)$ is a simple complex, one has $H_I^n(X) \simeq \mathrm{tot}(F_I^{-1} H^n F_I(X))$. We deduce from (5.5.10) the exact sequence in $\mathrm{C}(\mathscr{C})$, functorial with respect to X:

$$0 \to \mathrm{tot}(\tilde{\tau}_I^{\leq n-1}(X)) \to \mathrm{tot}(\tau_I^{\leq n}(X)) \to H_I^n(X)\,[-n] \to 0. \tag{5.6.4}$$

Theorem 5.6.1. *Let $f\colon X \to Y$ be a morphism of double complexes in $\mathscr{C}$, both satisfying (4.3.7). Assume that f induces an isomorphism $H_{II}H_I(X) \xrightarrow{\sim} H_{II}H_I(Y)$. Then $\mathrm{tot}(f)\colon \mathrm{tot}(X) \to \mathrm{tot}(Y)$ is a quasi-isomorphism.*

Proof. The hypothesis is equivalent to

(5.6.5) $H_I^n(f)\colon H_I^n(X) \to H_I^n(Y)$ is a qis for all n.

Since $H_I^n(\tau_I^{\geq p}X)$ is isomorphic to $H_I^n(X)$ for $n \geq p$ or to 0 otherwise, we get the isomorphisms

$$H_{II}H_I(\tau_I^{\geq p}X) \xrightarrow{\sim} H_{II}H_I(\tau_I^{\geq p}Y) \text{ for all } p.$$

For n fixed, we have

$$H^n(\mathrm{tot}(X)) \simeq H^n(\mathrm{tot}(\tau_I^{\geq p}X)) \text{ for } p \ll 0, \tag{5.6.6}$$

and similarly with Y instead of X. Hence, replacing X and Y with $\tau_I^{\geq p}X$ and $\tau_I^{\geq p}Y$, we may assume from the beginning that

$$X_I^n = 0 \text{ and } Y_I^n = 0 \text{ for } n \ll 0. \tag{5.6.7}$$

Using (5.6.4), we get a commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{tot}(\tilde{\tau}_I^{\leq n-1}(X)) \arrow[r] \arrow[d, "\mathrm{tot}(\tilde{\tau}_I^{\leq n-1}(f))"] & \mathrm{tot}(\tau_I^{\leq n}(X)) \arrow[r] \arrow[d, "\mathrm{tot}(\tau_I^{\leq n}(f))"] & H_I^n(X)\,[-n] \arrow[r] \arrow[d, "H_I^n(f)[-n]"] & 0 \\
0 \arrow[r] & \mathrm{tot}(\tilde{\tau}_I^{\leq n-1}(Y)) \arrow[r] & \mathrm{tot}(\tau_I^{\leq n}(Y)) \arrow[r] & H_I^n(Y)\,[-n] \arrow[r] & 0
\end{tikzcd} \tag{5.6.8}$$

By (5.6.5), the vertical arrow on the right is a qis for all $n \in \mathbb{Z}$. Thanks to (5.6.7), the vertical arrow on the left is a qis for $n \ll 0$. It follows by induction, using (5.6.3), that all vertical arrows are qis. Then the result follows from (5.6.6). □

Corollary 5.6.2. *Let X be a double complex in $\mathscr{C}$ satisfying (4.3.7). If $H_I(X) \simeq 0$, then $\mathrm{tot}(X)$ is qis to 0.*

Proof. Apply Theorem 5.6.1 with $Y = 0$ and use (5.6.5). ☐

Corollary 5.6.3. *Let X be a double complex in $\mathscr{C}$ satisfying (4.3.7). Assume that all rows $X^{j,\bullet}$ are exact for $j \neq n$. Then $\mathrm{tot}(X)$ is qis to $X^{n,\bullet}\,[-n]$.*

Proof. Denote by $\sigma_I^{\geq n}$ the "stupid" truncation functor which to a double complex X associates the double complex whose rows are those of X for $j \geq n$ and are 0 for $j < n$. Define similarly $\sigma_I^{\leq n}$. Now apply Theorem 5.6.1 to the morphism $\sigma_I^{\geq n}(X) \to X$, next to the morphism $\sigma_I^{\geq n}(X) \to \sigma_I^{\leq n}\sigma_I^{\geq n}(X) \simeq X^{n,\bullet}\,[-n]$. ☐

Corollary 5.6.4. *Let $X^{\bullet,\bullet}$ be a double complex. Assume that all rows $X^{j,\bullet}$ and columns $X^{\bullet,j}$ are 0 for $j < 0$ and are exact for $j > 0$. Then $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ for all p.*

Proof. Both $X^{0,\bullet}$ and $X^{\bullet,0}$ are qis to $\mathrm{tot}(X)$. ☐

Let us describe the isomorphism $H^p(X^{0,\bullet}) \simeq H^p(X^{\bullet,0})$ in the case where $\mathscr{C} = \mathrm{Mod}(A)$ by the so-called "Weil procedure".

Let $x^{p,0} \in X^{p,0}$, with $d'x^{p,0} = 0$ which represents $y \in H^p(X^{\bullet,0})$. Define $x^{p,1} = d''x^{p,0}$. Then $d'x^{p,1} = 0$, and the first column being exact, there exists $x^{p-1,1} \in X^{p-1,1}$ with $d'x^{p-1,1} = x^{p,1}$. One can iterate this procedure until getting $x^{0,p} \in X^{0,p}$. Since $d'd''x^{0,p} = 0$, and d' is injective on $X^{0,p}$ for $p > 0$ by the hypothesis, we get $d''x^{0,p} = 0$. The class of $x^{0,p}$ in $H^p(X^{0,\bullet})$ will be the image of y by the Weil procedure. Of course, one has to check that this image does not depend of the various choices we have made, and that it induces an isomorphism.

This can be visualized by the diagram:

$$\begin{tikzcd}
 & & & x^{0,p} \arrow[r, "d''"] \arrow[d, "d'"'] & 0 \\
 & & x^{1,p-2} \arrow[r, "d''"] \arrow[d, dotted] & x^{1,p-1} & \\
 & & {} & & \\
 & x^{p-1,1} \arrow[r, dotted] \arrow[d, "d'"'] & {} & & \\
x^{p,0} \arrow[r, "d''"] \arrow[d, "d'"'] & x^{p,1} & & & \\
0 & & & &
\end{tikzcd}$$

5.7 Koszul complexes

Recall that $\mathbf{k}$ denotes a commutative unital ring. In this section, we do not work in abstract abelian categories but in the category $\mathrm{Mod}(\mathbf{k})$.

If L is a finite free $\mathbf{k}$-module of rank n, one denotes by $\bigwedge^j L$ the $\mathbf{k}$-module consisting of j-multilinear alternate forms on the dual space L^* and calls it the j-th exterior power of L. (Recall that $L^* = \mathrm{Hom}_{\mathbf{k}}(L, \mathbf{k})$.)

Note that $\bigwedge^1 L \simeq L$ and $\bigwedge^n L \simeq \mathbf{k}$. One sets $\bigwedge^0 L = \mathbf{k}$.

If $(e_1, \dots, e_n)$ is a basis of L and $I = \{i_1 < \cdots < i_j\} \subset \{1, \dots, n\}$, one sets

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_j}.$$

For a subset $I \subset \{1, \dots, n\}$, one denotes by $|I|$ its cardinal. Recall that:

$$\bigwedge^{j} L \text{ is free with basis } \{e_I; I \subset \{1, \dots, n\}, |I| = j\}.$$

If $i_1, \dots, i_m$ belong to the set $(1, \dots, n)$, one defines $e_{i_1} \wedge \cdots \wedge e_{i_m}$ by reducing to the case where $i_1 < \cdots < i_j$, using the convention $e_i \wedge e_j = -e_j \wedge e_i$.

Let M be a $\mathbf{k}$-module and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be n $\mathbf{k}$-linear endomorphisms of M which commute with one another:

$$[\varphi_i, \varphi_j] = 0, \ 1 \leq i, j \leq n.$$

(Recall the notation $[a, b] := ab - ba$.) Set

$$M^{(j)} = M \otimes \bigwedge^{j} \mathbf{k}^n.$$

Hence $M^{(0)} = M$ and $M^{(n)} \simeq M$. Denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbf{k}^n$. Hence, any element of $M^{(j)}$ may be written uniquely as a sum

$$m = \sum_{|I|=j} m_I \otimes e_I.$$

One defines $d \in \mathrm{Hom}_{\mathbf{k}}(M^{(j)}, M^{(j+1)})$ by:

$$d(m \otimes e_I) = \sum_{i=1}^{n} \varphi_i(m) \otimes e_i \wedge e_I$$

and extending d by $\mathbf{k}$-linearity. Using the commutativity of the φ_i's one checks easily that $d \circ d = 0$. Hence we get a complex, called a Koszul complex and denoted $K^\bullet(M, \varphi)$:

$$0 \to M^{(0)} \xrightarrow{d} \cdots \to M^{(n)} \to 0.$$

When $n = 1$, the cohomology of this complex gives the kernel and cokernel of φ_1. More generally,

$$\begin{aligned} H^0(K^\bullet(M, \varphi)) &\simeq \ker \varphi_1 \cap \ldots \cap \ker \varphi_n, \\ H^n(K^\bullet(M, \varphi)) &\simeq M/(\varphi_1(M) + \cdots + \varphi_n(M)). \end{aligned}$$

Set $\varphi' = \{\varphi_1, \dots, \varphi_{n-1}\}$ and denote by d' the differential in $K^\bullet(M, \varphi')$. Then φ_n defines a morphism

$$\widetilde{\varphi}_n : K^\bullet(M, \varphi') \to K^\bullet(M, \varphi') \tag{5.7.1}$$

Lemma 5.7.1. *The complex $K^\bullet(M, \varphi)[1]$ is isomorphic to the mapping cone of $-\widetilde{\varphi}_n$.*

Proof. [3] Consider the diagram

$$\begin{tikzcd}
\mathrm{Mc}(\widetilde{\varphi}_n)^p \arrow[r, "d_M^p"'] \arrow[d, "\lambda^p"'] & \mathrm{Mc}(\widetilde{\varphi}_n)^{p+1} \arrow[d, "\lambda^{p+1}"'] \\
K^{p+1}(M, \varphi) \arrow[r, "d_K^{p+1}"'] & K^{p+2}(M, \varphi)
\end{tikzcd}$$

---

[3] The proof may be skipped

given explicitly by:

$$\begin{tikzcd}
(M\otimes\bigwedge^{p+1}\mathbf{k}^{n-1})\oplus(M\otimes\bigwedge^{p}\mathbf{k}^{n-1}) \arrow[r, "\begin{pmatrix}-d' & 0\\ -\varphi_n & d'\end{pmatrix}"'] \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] & (M\otimes\bigwedge^{p+2}\mathbf{k}^{n-1})\oplus(M\otimes\bigwedge^{p+1}\mathbf{k}^{n-1}) \arrow[d, "\mathrm{id}\oplus(\mathrm{id}\otimes e_n\wedge)"'] \\
M\otimes\bigwedge^{p+1}\mathbf{k}^{n} \arrow[r, "-d"'] & M\otimes\bigwedge^{p+2}\mathbf{k}^{n}
\end{tikzcd}$$

Then

$$\begin{aligned}
&d_M^p(a\otimes e_J+b\otimes e_K)=-d'(a\otimes e_J)+(d'(b\otimes e_K)-\varphi_n(a)\otimes e_J),\\
&\lambda^p(a\otimes e_J+b\otimes e_K)=a\otimes e_J+b\otimes e_n\wedge e_K.
\end{aligned}$$

(i) The vertical arrows are isomorphisms. Indeed, let us treat the first one. It is described by:

$$\sum_J a_J\otimes e_J+\sum_K b_K\otimes e_K\mapsto\sum_J a_J\otimes e_J+\sum_K b_K\otimes e_n\wedge e_K \tag{5.7.2}$$

with $|J|=p+1$ and $|K|=p$. Any element of $M\otimes\bigwedge^{p+1}\mathbf{k}^n$ may uniquely be written as in the right hand side of (5.7.2).

(ii) The diagram commutes. Indeed,

$$\begin{aligned}
\lambda^{p+1}\circ d_M^p(a\otimes e_J+b\otimes e_K)&=-d'(a\otimes e_J)+e_n\wedge d'(b\otimes e_K)-\varphi_n(a)\otimes e_n\wedge e_J\\
&=-d'(a\otimes e_J)-d'(b\otimes e_n\wedge e_K)-\varphi_n(a)\otimes e_n\wedge e_J,\\
d_K^{p+1}\circ\lambda^p(a\otimes e_J+b\otimes e_K)&=-d(a\otimes e_J+b\otimes e_n\wedge e_K)\\
&=-d'(a\otimes e_J)-\varphi_n(a)\otimes e_n\wedge e_J-d'(b\otimes e_n\wedge e_K).
\end{aligned}$$

☐

Theorem 5.7.2. *There exists a* $\mathbf{k}$*-linear long exact sequence*

$$\cdots\to H^j(K^\bullet(M,\varphi'))\xrightarrow{\varphi_n}H^j(K^\bullet(M,\varphi'))\to H^{j+1}(K^\bullet(M,\varphi))\to\cdots \tag{5.7.3}$$

Proof. Apply Lemma 5.7.1 and the long exact sequence (5.5.15). ☐

Definition 5.7.3.
(i) If for each j, $1\le j\le n$, φ_j is injective as an endomorphism of $M/(\varphi_1(M)+\cdots+\varphi_{j-1}(M))$, one says that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence.

(ii) If for each j, $1\le j\le n$, φ_j is surjective as an endomorphism of $\ker\varphi_1\cap\ldots\cap\ker\varphi_{j-1}$, one says that $(\varphi_1,\dots,\varphi_n)$ is a coregular sequence.

Corollary 5.7.4.
(i) *If* $(\varphi_1,\dots,\varphi_n)$ *is a regular sequence, then* $H^j(K^\bullet(M,\varphi))\simeq 0$ *for* $j\neq n$.

(ii) *If* $(\varphi_1,\dots,\varphi_n)$ *is a coregular sequence, then* $H^j(K^\bullet(M,\varphi))\simeq 0$ *for* $j\neq 0$.

Proof. Assume for example that $(\varphi_1,\dots,\varphi_n)$ is a regular sequence and let us argue by induction on n. The cohomology of $K^\bullet(M,\varphi')$ is thus concentrated in degree $n-1$ and is isomorphic to $M/(\varphi_1(M)+\cdots+\varphi_{n-1}(M))$. By the hypothesis, φ_n is injective on this group, and Corollary 5.7.4 follows. ☐

Second proof. Let us give a direct proof of the Corollary in case $n = 2$ for coregular sequences. Hence we consider the complex:

$$0 \to M \xrightarrow{d} M \times M \xrightarrow{d} M \to 0$$

where $d(x) = (\varphi_1(x), \varphi_2(x))$, $d(y, z) = \varphi_2(y) - \varphi_1(z)$ and we assume φ_1 is surjective on M, φ_2 is surjective on $\ker \varphi_1$.

Let $(y, z) \in M \times M$ with $\varphi_2(y) = \varphi_1(z)$. We look for $x \in M$ solution of $\varphi_1(x) = y, \quad \varphi_2(x) = z$. First choose $x' \in M$ with $\varphi_1(x') = y$. Then $\varphi_2 \circ \varphi_1(x') = \varphi_2(y) = \varphi_1(z) = \varphi_1 \circ \varphi_2(x')$. Thus $\varphi_1(z - \varphi_2(x')) = 0$ and there exists $t \in M$ with $\varphi_1(t) = 0, \quad \varphi_2(t) = z - \varphi_2(x')$. Hence $y = \varphi_1(t + x'), \quad z = \varphi_2(t + x')$ and $x = t + x'$ is a solution to our problem. □

Example 5.7.5. Let $\mathbf{k}$ be a field of characteristic 0 and let $A = \mathbf{k}[x_1, \ldots, x_n]$.
(i) Denote by $x_i\cdot$ the multiplication by x_i in A. We get the complex:

$$0 \to A^{(0)} \xrightarrow{d} \cdots \to A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I x_j \cdot a_I \otimes e_j \wedge e_I.$$

The sequence $(x_1\cdot, \ldots, x_n\cdot)$ is a regular sequence. Hence the Koszul complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}$.
(ii) Denote by ∂_i the partial derivation with respect to x_i. This is a $\mathbf{k}$-linear map on the $\mathbf{k}$-vector space A. Hence we get a Koszul complex

$$0 \to A^{(0)} \xrightarrow{d} \cdots \xrightarrow{d} A^{(n)} \to 0$$

where:

$$d(\sum_I a_I \otimes e_I) = \sum_{j=1}^{n} \sum_I \partial_j(a_I) \otimes e_j \wedge e_I.$$

The sequence $(\partial_1\cdot, \ldots, \partial_n\cdot)$ is a coregular sequence and the above complex is exact except in degree 0 where its cohomology is isomorphic to $\mathbf{k}$. Writing dx_j instead of e_j, we recognize the “de Rham complex”.

Example 5.7.6. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x, y]$, $M = \mathbf{k} \simeq A/xA + yA$. Let us calculate a free (hence, projective) resolution of M. Since (x, y) is a regular sequence of endomorphisms of A (viewed as a $\mathbf{k}$-module), M is quasi-isomorphic to the complex:

$$M^\bullet : 0 \to A \xrightarrow{u} A^2 \xrightarrow{v} A \to 0,$$

where $u(a) = (ya, -xa)$, $v(b, c) = xb + yc$ and the module A on the right stands in degree 0. Therefore, for N an A-module, $\mathrm{Hom}_A(M^\bullet, N)$ is represented by the complex:

$$0 \to N \xrightarrow{v'} N^2 \xrightarrow{u'} N \to 0,$$

where $v' = \mathrm{Hom}\,(v, N)$, $u' = \mathrm{Hom}\,(u, N)$ and the module N on the left stands in degree 0. Since $v'(n) = (xn, yn)$ and $u'(m, l) = ym - xl$, we find again a Koszul complex. Choosing $N = A$, its cohomology is concentrated in degree 2 and isomorphic to $\mathbf{k}$.

Example 5.7.7. Let $W = W_n(\mathbf{k})$ be the Weyl algebra introduced in Example 1.2.2, and denote by $\cdot\partial_i$ the multiplication on the right by ∂_i. Then $(\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence on W and we get the Koszul complex:

$$0 \to W^{(0)} \xrightarrow{\delta} \cdots \to W^{(n)} \to 0$$

where:

$$\delta(\sum_I a_I \otimes e_I) = \sum_{j=1}^n \sum_I a_I \cdot \partial_j \otimes e_j \wedge e_I.$$

This complex is exact except in degree n where its cohomology is isomorphic to $\mathbf{k}[x]$ (see Exercise 5.9).

Remark 5.7.8. One may also encounter co-Koszul complexes. For $I = (i_1, \dots, i_k)$, introduce

$$e_j \lfloor e_I = \begin{cases} 0 & \text{if } j \notin \{i_1, \dots, i_k\} \\ (-1)^{l+1} e_{I_{\hat{l}}} := (-1)^{l+1} e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k} & \text{if } e_{i_l} = e_j \end{cases}$$

where $e_{i_1} \wedge \dots \wedge \widehat{e_{i_l}} \wedge \dots \wedge e_{i_k}$ means that e_{i_l} should be omitted in $e_{i_1} \wedge \dots \wedge e_{i_k}$. Define δ by:

$$\delta(m \otimes e_I) = \sum_{j=1}^n \varphi_j(m) e_j \lfloor e_I.$$

Here again one checks easily that $\delta \circ \delta = 0$, and we get the complex:

$$K_\bullet(M, \varphi) : 0 \to M^{(n)} \xrightarrow{\delta} \cdots \to M^{(0)} \to 0,$$

This complex is in fact isomorphic to a Koszul complex. Consider the isomorphism

$$*\colon \bigwedge^j \mathbf{k}^n \xrightarrow{\sim} \bigwedge^{n-j} \mathbf{k}^n$$

which associates $\varepsilon_I m \otimes e_{\hat{I}}$ to $m \otimes e_I$, where $\hat{I} = (1, \dots, n) \setminus I$ and ε_I is the signature of the permutation which sends $(1, \dots, n)$ to $I \sqcup \hat{I}$ (any $i \in I$ is smaller than any $j \in \hat{I}$). Then, up to a sign, $*$ interchanges d and δ.

De Rham complexes

Let E be a real vector space of dimension n and let U be an open subset of E. Denote as usual by $\mathscr{C}^\infty(U)$ the $\mathbb{C}$-algebra of $\mathbb{C}$-valued functions on U of class C^∞. Recall that $\Omega^1(U)$ denotes the $\mathscr{C}^\infty(U)$-module of C^∞-functions on U with values in $E^* \otimes_\mathbb{R} \mathbb{C} \simeq \mathrm{Hom}_\mathbb{R}(E, \mathbb{C})$. Hence

$$\Omega^1(U) \simeq E^* \otimes_\mathbb{R} \mathscr{C}^\infty(U).$$

For $p \in \mathbb{N}$, one sets

$$\begin{aligned} \Omega^p(U) &:= \bigwedge^p \Omega^1(U) \\ &\simeq (\bigwedge^p E^*) \otimes_\mathbb{R} \mathscr{C}^\infty(U). \end{aligned}$$

(The first exterior product is taken over the commutative ring $\mathscr{C}^\infty(U)$ and the second one over $\mathbb{R}$.) Hence, $\Omega^0(U) = \mathscr{C}^\infty(U)$, $\Omega^p(U) = 0$ for $p > n$ and $\Omega^n(U)$ is free of rank 1 over $\mathscr{C}^\infty(U)$. The differential is a $\mathbb{C}$-linear map

$$d\colon \mathscr{C}^\infty(U) \to \Omega^1(U).$$

The differential extends by multilinearity as a $\mathbb{C}$-linear map $d\colon \Omega^p(U) \to \Omega^{p+1}(U)$ satisfying

$$\begin{cases} d^2 = 0, \\ d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-)^p \omega_1 \wedge d\omega_2 \text{ for any } \omega_1 \in \Omega^p(U). \end{cases} \tag{5.7.4}$$

We get a complex, called the De Rham complex, that we denote by $\mathrm{DR}(U)$:

$$\mathrm{DR}(U) := 0 \to \Omega^0(U) \xrightarrow{d} \cdots \to \Omega^n(U) \to 0. \tag{5.7.5}$$

Let us choose a basis $(e_1, \dots, e_n)$ of E and denote by x_i the function which, to $x = \sum_{i=1}^n x_i \cdot e_i \in E$, associates its i-th cordinate x_i. Then $(dx_1, \dots, dx_n)$ is the dual basis on E^* and the differential of a function φ is given by

$$d\varphi = \sum_{i=1}^n \partial_i \varphi \, dx_i.$$

where $\partial_i \varphi := \dfrac{\partial \varphi}{\partial x_i}$. By its construction, the Koszul complex of $(\partial_1, \dots, \partial_n)$ acting on $\mathscr{C}^\infty(U)$ is nothing but the De Rham complex:

$$K^\bullet(\mathscr{C}^\infty(U), (\partial_1, \dots, \partial_n)) = \mathrm{DR}(U).$$

Note that $H^0(\mathrm{DR}(U))$ is the space of locally constant functions on U, and therefore is isomorphic to $\mathbb{C}^{\# cc(U)}$ where $\# cc(U)$ denotes the cardinal of the set of connected components of U. Using sheaf theory, one proves that all cohomology groups $H^j(\mathrm{DR}(U))$ are topological invariants of U.

Holomorphic De Rham complexes

Replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, $\mathscr{C}^\infty(U)$ with $\mathscr{O}(U)$, the space of holomorphic functions on U and the real derivation with the holomorphic derivation, one constructs similarly the holomorphic De Rham complex.

Example 5.7.9. Let $n = 1$ and let $U = \mathbb{C} \setminus \{0\}$. The holomorphic De Rham complex reduces to

$$0 \to \mathscr{O}(U) \xrightarrow{\partial_z} \mathscr{O}(U) \to 0.$$

Its cohomology is isomorphic to $\mathbb{C}$ in degree 0 and 1.

Exercises to Chapter 5

Exercise 5.1. Prove assertion (iv) in Proposition 5.2.3, that is, prove that for a ring A and a small set I, the two functors $\prod$ and $\bigoplus$ from $\mathrm{Fct}(I, \mathrm{Mod}(A))$ to $\mathrm{Mod}(A)$ are exact.

Exercise 5.2. Consider two complexes in an abelian category $\mathscr{C}$: $X_1' \to X_1 \to X_1''$ and $X_2' \to X_2 \to X_2''$. Prove that the two sequences are exact if and only if the sequence $X_1' \oplus X_2' \to X_1 \oplus X_2 \to X_1'' \oplus X_2''$ is exact.

Exercise 5.3. Let A be a ring.

(i) Prove that a free module is projective.

(ii) Prove that a module P is projective if and only if it is a direct summand of a free module (*i.e.*, there exists a module K such that $P \oplus K$ is free).

(iii) An A-module M is flat if the functor $\bullet \otimes_A M$ is exact. (One defines similarly flat right A-modules.) Deduce from (ii) that projective modules are flat.

Exercise 5.4. (see [God58, Th. 1.2.2]) If M is a $\mathbb{Z}$-module, set $M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$.

(i) Prove that $\mathbb{Q}/\mathbb{Z}$ is injective in $\mathrm{Mod}(\mathbb{Z})$.

(ii) Prove that the map $\mathrm{Hom}_{\mathbb{Z}}(M, N) \to \mathrm{Hom}_{\mathbb{Z}}(N^\vee, M^\vee)$ is injective for any $M, N \in \mathrm{Mod}(\mathbb{Z})$.

(iii) Prove that if P is a right projective A-module, then $P^\vee$ is left A-injective.

(iv) Let M be an A-module. Prove that there exists an injective A-module I and a monomorphism $M \to I$.

(Hint: (iii) Use formula (1.2.4).

(iv) Prove that $M \mapsto M^{\vee\vee}$ is an injective map using (ii), and replace M with $M^{\vee\vee}$.)

Exercise 5.5. Let $\mathscr{C}$ be an abelian category which admits small colimits and such that small filtered colimits are exact. Let $\{X_i\}_{i \in I}$ be a family of objects of $\mathscr{C}$ indexed by a small set I and let $i_0 \in I$. Prove that the natural morphism $X_{i_0} \to \bigoplus_{i \in I} X_i$ is a monomorphism.

Exercise 5.6. Let $\mathscr{C}$ be an abelian category.

(i) Prove that a complex $0 \to X \to Y \to Z$ is exact iff and only if for any object $W \in \mathscr{C}$ the complex of abelian groups $0 \to \mathrm{Hom}_{\mathscr{C}}(W, X) \to \mathrm{Hom}_{\mathscr{C}}(W, Y) \to \mathrm{Hom}_{\mathscr{C}}(W, Z)$ is exact.

(ii) By reversing the arrows, state and prove a similar statement for a complex $X \to Y \to Z \to 0$.

Exercise 5.7. Recall Definition 2.4.1. Let $\mathscr{C}$ be an abelian category and consider a commutative diagram:

$$\begin{tikzcd}
V \arrow[r, "f'"] \arrow[d, "g'"] & Y \arrow[d, "g"] \\
X \arrow[r, "f"] & Z.
\end{tikzcd}$$

The square is Cartesian if the sequence $0 \to V \to X \times Y \to Z$ is exact, that is, if $V \simeq X \times_Z Y$ (recall that $X \times_Z Y = \ker(f - g)$, where $f - g : X \oplus Y \to Z$). The square is co-Cartesian if the sequence $V \to X \oplus Y \to Z \to 0$ is exact, that is, if $Z \simeq X \oplus_V Y$ (recall that $X \oplus_Z Y = \operatorname{Coker}(f' - g')$, where $f' - g' : V \to X \times Y$).

(i) Assume the square is Cartesian and f is an epimorphism. Prove that f' is an epimorphism.

(ii) Assume the square is co-Cartesian and f' is a monomorphism. Prove that f is a monomorphism.

Exercise 5.8. Let $\mathscr{C}$ be an abelian category and consider a double complexe

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & X'_0 \arrow[r] \arrow[d] & X_0 \arrow[r] \arrow[d] & X''_0 \arrow[d] \\
0 \arrow[r] & X'_1 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & X''_1 \arrow[d] \\
0 \arrow[r] & X'_2 \arrow[r] & X_2 \arrow[r] & X''_2
\end{tikzcd}$$

Assume that all rows are exact as well as the second and third column. Prove that all columns are exact.

Exercise 5.9. Let $\mathbf{k}$ be a field of characteristic 0, $W := W_n(\mathbf{k})$ the Weyl algebra in n variables.

(i) Denote by $x_i\cdot : W \to W$ the multiplication on the left by x_i on W (hence, the $x_i\cdot$'s are morphisms of right W-modules). Prove that $\varphi = (x_1\cdot, \dots, x_n\cdot)$ is a regular sequence and calculate $H^j(K^\bullet(W, \varphi))$.

(ii) Denote $\cdot\partial_i$ the multiplication on the right by ∂_i on W. Prove that $\psi = (\cdot\partial_1, \dots, \cdot\partial_n)$ is a regular sequence and calculate $H^j(K^\bullet(W, \psi))$.

(iii) Now consider the left $W_n(\mathbf{k})$-module $\mathscr{O} := \mathbf{k}[x_1, \dots, x_n]$ and the $\mathbf{k}$-linear map $\partial_i : \mathscr{O} \to \mathscr{O}$ (derivation with respect to x_i). Prove that $\lambda = (\partial_1, \dots, \partial_n)$ is a coregular sequence and calculate $H^j(K^\bullet(\mathscr{O}, \lambda))$.

Exercise 5.10. Let $A = W_2(\mathbf{k})$ be the Weyl algebra in two variables. Construct the Koszul complex associated to $\varphi_1 = \cdot x_1$, $\varphi_2 = \cdot\partial_2$ and calculate its cohomology.

Exercise 5.11. Let $\mathbf{k}$ be a field, $A = \mathbf{k}[x, y]$ and consider the A-module $M = \bigoplus_{i \geq 1} \mathbf{k}[x]t^i$, where the action of $x \in A$ is the usual one and the action of $y \in A$ is defined by $y \cdot x^n t^{j+1} = x^n t^j$ for $j \geq 1$, $y \cdot x^n t = 0$. Define the endomorphisms of M, $\varphi_1(m) = x \cdot m$ and $\varphi_2(m) = y \cdot m$. Calculate the cohomology of the Kozsul complex $K^\bullet(M, \varphi)$.

Exercise 5.12. Let $\mathscr{C}$ be an abelian actegory which admits small direct sums and let I be a small set. For $X, Y \in \mathscr{C}$, prove the isomorphism

$$\operatorname{Hom}_{\mathbf{Set}}(I, \operatorname{Hom}_{\mathscr{C}}(Y, X)) \simeq \operatorname{Hom}_{\mathscr{C}}(Y^{\oplus I}, X).$$

(Hint: see (1.1.3) and (1.1.5).)

Exercise 5.13. Let $F : \mathscr{C} \to \mathscr{C}'$ be an additive functor of abelian categories. Prove that if F is faithful then it is conservative. Conversely, assume that F is conservative and either left or right exact. Prove that F is faithful.

Exercise 5.14. Let $\mathscr{C}$ be an abelian category which admits small coproducts and a cogenerator G. Let $f\colon X \to Y$ be a morphism in $\mathscr{C}$ and assume that $\mathrm{Hom}_{\mathscr{C}}(G, X) \to \mathrm{Hom}_{\mathscr{C}}(G, Y)$ is surjective. Prove that f is an epimorphism.

(Hint: use Lemma 5.4.2 and Exercise 1.6.)

Chapter 6

Triangulated categories

Triangulated categories play an iimportant role in mathematics and this subject would deserve more that the short chapter that we present here. They are a substitute, in some sense, to abelian categories, the distinguished triangles playing the role of the exact sequences, and they are naturally associated to additive categories. Indeed, as we shall see the homotopy category $\mathrm{K}(\mathscr{C})$ associated with an additive category $\mathscr{C}$ is naturally triangulated.

We have restricted ourselves to describe the main properties of such categories as well as some basic results without addressing many related topics. In particular, we localize triangulated categories and triangulated functors with the construction of derived categories in mind.

Some tedious proofs are skipped, referring to [KS06].

Note that the fact that the morphism in TR4 (see below) is not unique is the source of many troubles. It is the main obstacle encountered when trying to "glue" derived categories. This difficulty is overcome with the theory of ∞-categories where stable categories play the role of triangulated categories.

Some references: [GM96, KS90, KS06, Nee01, Ver96, Wei94, Yek20].

6.1 Triangulated categories

Definition 6.1.1. A category with translation $(\mathscr{D}, T)$ is an additive category $\mathscr{D}$ endowed with an automorphism $T\colon \mathscr{D} \to \mathscr{D}$ (i.e., an invertible functor), called the translation functor.

A triangle in $(\mathscr{D}, T)$ is a sequence of morphisms:

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X). \tag{6.1.1}$$

A morphism of triangles is a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X').
\end{tikzcd}$$

Example 6.1.2. The triangle $X \xrightarrow{f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is isomorphic to the triangle (6.1.1), but the triangle $X \xrightarrow{-f} Y \xrightarrow{-g} Z \xrightarrow{-h} T(X)$ is not isomorphic to the triangle (6.1.1) in general.

Definition 6.1.3. A triangulated category is an additive category $\mathscr{D}$ endowed with an automorphism T and a family of triangles called distinguished triangles (d.t. for short), this family satisfying axioms TR0 - TR5 below.

TR0 A triangle isomorphic to a d.t. is a d.t.

TR1 The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is a d.t.

TR2 For all $f\colon X \to Y$ there exists a d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$.

TR3 A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ is a d.t. if and only if $Y \xrightarrow{g} Z \xrightarrow{h} T(X) \xrightarrow{-T(f)} T(Y)$ is a d.t.

TR4 Given two d.t. $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ and $X' \xrightarrow{f'} Y' \xrightarrow{g'} Z' \xrightarrow{h'} T(X')$ and morphisms $\alpha\colon X \to X'$ and $\beta : Y \to Y'$ with $f' \circ \alpha = \beta \circ f$, there exists a morphism $\gamma\colon Z \to Z'$ giving rise to a morphism of d.t.:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, dotted, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X'),
\end{tikzcd}$$

TR5 (Octahedral axiom) Given three d.t.

$$X \xrightarrow{f} Y \xrightarrow{h} Z' \to T(X),$$
$$Y \xrightarrow{g} Z \xrightarrow{k} X' \to T(Y),$$
$$X \xrightarrow{g\circ f} Z \xrightarrow{l} Y' \to T(X),$$

there exists a distinguished triangle $Z' \xrightarrow{\varphi} Y' \xrightarrow{\psi} X' \to T(Z')$ making the diagram below commutative:

(6.1.2)
$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathrm{id}"'] & Y \arrow[r, "h"] \arrow[d, "g"'] & Z' \arrow[r] \arrow[d, dotted, "\varphi"'] & T(X) \arrow[d, "\mathrm{id}"'] \\
X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \arrow[r, "l"] \arrow[d, "\mathrm{id}"'] & Y' \arrow[r] \arrow[d, dotted, "\psi"'] & T(X) \arrow[d, "T(f)"'] \\
Y \arrow[r, "g"] \arrow[d, "h"'] & Z \arrow[r, "k"] \arrow[d, "l"'] & X' \arrow[r] \arrow[d, "\mathrm{id}"'] & T(Y) \arrow[d, "T(h)"'] \\
Z' \arrow[r, dotted, "\varphi"] & Y' \arrow[r, dotted, "\psi"] & X' \arrow[r, dotted] & T(Z')
\end{tikzcd}$$

Diagram (6.1.2) is often called the octahedron diagram. Indeed, it can be written using the vertexes of an octahedron.

$$\begin{tikzcd}
 & Y' \arrow[dr, dotted, "\psi"] \arrow[ddl, dashed] & \\
Z' \arrow[ur, dotted, "\varphi"] \arrow[d, "+1"'] & & X' \arrow[ll, "+1"'] \arrow[ddl] \\
X \arrow[rr, dashed, "+1"] \arrow[dr, "f"'] & & Z \arrow[u] \arrow[uul, dashed] \\
 & Y \arrow[uul] \arrow[ur, "g"'] &
\end{tikzcd}$$

In this diagram, the notation $A \xrightarrow{+1} B$ means $A \to T(B)$.

Remark 6.1.4. The category $\mathscr{D}^{\mathrm{op}}$ endowed with the image by the contravariant functor $\mathrm{op}\colon \mathscr{D} \to \mathscr{D}^{\mathrm{op}}$ of the family of the d.t. in $\mathscr{D}$, is a triangulated category.

6.2 Triangulated and cohomological functors

Definition 6.2.1. (i) A triangulated functor of triangulated categories $F\colon (\mathscr{D}, T) \to (\mathscr{D}', T')$ is an additive functor which satisfies $F \circ T \simeq T' \circ F$ and which sends distinguished triangles to distinguished triangles.

(ii) A triangulated subcategory $\mathscr{D}'$ of $\mathscr{D}$ is a subcategory $\mathscr{D}'$ of $\mathscr{D}$ which is triangulated and such that the functor $\mathscr{D}' \to \mathscr{D}$ is triangulated.

(iii) Let $(\mathscr{D}, T)$ be a triangulated category, $\mathscr{C}$ an abelian category, $F\colon \mathscr{D} \to \mathscr{C}$ an additive functor. One says that F is a cohomological functor if for any d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathscr{C}$.

Remark 6.2.2. By TR3, a cohomological functor gives rise to a long exact sequence:

$$\cdots \to F(X) \to F(Y) \to F(Z) \to F(T(X)) \to \cdots \tag{6.2.1}$$

Proposition 6.2.3. (i) *If $X \xrightarrow{f} Y \xrightarrow{g} Z \to T(X)$ is a d.t. then $g \circ f = 0$.*

(ii) *For any $W \in \mathscr{D}$, the functors $\mathrm{Hom}_{\mathscr{D}}(W, \cdot)$ and $\mathrm{Hom}_{\mathscr{D}}(\cdot, W)$ are cohomological.*

Note that (ii) means that if $\varphi : W \to Y$ (resp. $\varphi\colon Y \to W$) satisfies $g \circ \varphi = 0$ (resp. $\varphi \circ f = 0$), then φ factorizes through f (resp. through g).

Proof. (i) Applying TR1 and TR4 we get a commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(X) \arrow[d, "\mathrm{id}"'] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X).
\end{tikzcd}$$

Then $g \circ f$ factorizes through 0.

(ii) Let $X \to Y \to Z \to T(X)$ be a d.t. and let $W \in \mathscr{D}$. We want to show that

$$\mathrm{Hom}\,(W,X) \xrightarrow{f\circ} \mathrm{Hom}\,(W,Y) \xrightarrow{g\circ} \mathrm{Hom}\,(W,Z)$$

is exact, i.e., for all $\varphi\colon W \to Y$ such that $g \circ \varphi = 0$, there exists $\psi\colon W \to X$ such that $\varphi = f \circ \psi$. This means that the dotted arrow below may be completed, and this follows from the axioms TR4 and TR3.

$$\begin{tikzcd}
W \arrow[r, "\mathrm{id}"] \arrow[d, dotted] & W \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[r] \arrow[d] & T(W) \arrow[d, dotted] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & T(X).
\end{tikzcd}$$

The proof for $\mathrm{Hom}\,(\cdot, W)$ is similar. $\square$

Proposition 6.2.4. *Consider a morphism of d.t.:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X').
\end{tikzcd}$$

If α and β are isomorphisms, then so is γ.

Proof. Apply $\mathrm{Hom}\,(W, \cdot)$ to this diagram and write $\tilde{X}$ instead of $\mathrm{Hom}\,(W,X)$, $\tilde{\alpha}$ instead of $\mathrm{Hom}\,(W,\alpha)$, etc. We get the commutative diagram:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{\alpha}"'] & \tilde{Y} \arrow[r, "\tilde{g}"] \arrow[d, "\tilde{\beta}"'] & \tilde{Z} \arrow[r, "\tilde{h}"] \arrow[d, "\tilde{\gamma}"'] & \widetilde{T(X)} \arrow[d, "\widetilde{T(\alpha)}"'] \\
\tilde{X}' \arrow[r, "\tilde{f}'"] & \tilde{Y}' \arrow[r, "\tilde{g}'"] & \tilde{Z}' \arrow[r, "\tilde{h}'"] & \widetilde{T(X')}.
\end{tikzcd}$$

The rows are exact in view of the preceding proposition and $\tilde{\alpha}$, $\tilde{\beta}$, $\widetilde{T(\alpha)}$, $\widetilde{T(\beta)}$ are isomorphisms. Therefore $\tilde{\gamma} = \mathrm{Hom}\,(W,\gamma) : \mathrm{Hom}\,(W,Z) \to \mathrm{Hom}\,(W,Z')$ is an isomorphism. This implies that γ is an isomorphism by the Yoneda lemma. $\square$

Corollary 6.2.5. *Let $\mathscr{D}'$ be a full triangulated category of $\mathscr{D}$.*

(i) *Consider a triangle $X \xrightarrow{f} Y \to Z \to T(X)$ in $\mathscr{D}'$ and assume that this triangle is distinguished in $\mathscr{D}$. Then it is distinguished in $\mathscr{D}'$.*

(ii) *Consider a d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$, with X and Y in $\mathscr{D}'$. Then there exists $Z' \in \mathscr{D}'$ and an isomorphism $Z \simeq Z'$.*

Proof. (i) There exists a d.t. $X \xrightarrow{f} Y \to Z' \to T(X)$ in $\mathscr{D}'$. Then Z' is isomorphic to Z by TR4 and Proposition 6.2.4.
(ii) Apply TR2 to the morphism $X \to Y$ in $\mathscr{D}'$. $\square$

Remark 6.2.6. The proof of Proposition 6.2.4 does not make use of axiom TR 5, and this proposition implies that TR 5 is equivalent to the axiom:
TR5': given $f\colon X \to Y$ and $g\colon Y \to Z$, there exists a commutative diagram (6.1.2) such that all rows are d.t.

Remark 6.2.7. (a) The morphism γ in TR 4 is not unique and this is the origin of many troubles.

(b) Similarly, it follows from Proposition 6.2.4 that the object Z given in TR2 is unique up to isomorphism. However, this isomorphism is not unique, and again this is the source of many troubles (e.g., glueing problems in sheaf theory).

6.3 Applications to the homotopy category

Let $\mathscr{C}$ be an additive category. Both $\mathrm{C}(\mathscr{C})$ and $\mathrm{K}(\mathscr{C})$ are endowed with a natural translation functor. (Recall that the homotopy category $\mathrm{K}(\mathscr{C})$ is defined by identifying to zero the morphisms in $\mathrm{C}(\mathscr{C})$ homotopic to zero.)

Also recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one defines its mapping cone $\mathrm{Mc}(f)$, an object of $\mathrm{C}(\mathscr{C})$, and there is a natural triangle

$$Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{f[1]} Y[1]. \tag{6.3.1}$$

Such a triangle is called a mapping cone triangle. Clearly, a triangle in $\mathrm{C}(\mathscr{C})$ gives rise to a triangle in the homotopy category $\mathrm{K}(\mathscr{C})$.

Definition 6.3.1. A distinguished triangle (d.t. for short) in $\mathrm{K}(\mathscr{C})$ is a triangle isomorphic in $\mathrm{K}(\mathscr{C})$ to a mapping cone triangle.

Theorem 6.3.2. *The category* $\mathrm{K}(\mathscr{C})$ *endowed with the shift functor* [1] *and the family of d.t. is a triangulated category.*

We shall not give here the proof of this classical and fundamental result, referring to [KS06, Th. 11.2.6].

Notation 6.3.3. We shall often write $X \to Y \to Z \xrightarrow{+1}$ instead of $X \to Y \to Z \to X[1]$ to denote a d.t. in $\mathrm{K}(\mathscr{C})$.

6.4 Localization of triangulated categories

Definition 6.4.1. A *null system* $\mathscr{N}$ in $\mathscr{D}$ is a full triangulated saturated subcategory of $\mathscr{D}$.

A null system $\mathscr{N}$ satisfies:

N1 $0 \in \mathscr{N}$,

N2 $X \in \mathscr{N}$ if and only if $T(X) \in \mathscr{N}$,

N3 if $X \to Y \to Z \to T(X)$ is a d.t. in $\mathscr{D}$ and $X, Y \in \mathscr{N}$ then $Z \in \mathscr{N}$.

One easily checks that if $\mathscr{N}$ is a full saturated subcategory of $\mathscr{D}$ satisfying N1-N2-N3, then the restriction of T to $\mathscr{N}$ and the family of d.t. $X \to Y \to Z \to T(X)$ in $\mathscr{D}$ with $X, Y, Z \in \mathscr{N}$ make $\mathscr{N}$ a null system of $\mathscr{D}$. Moreover, it has the property that given a d.t. as above in $\mathscr{D}$, the three objects X, Y, Z belong to $\mathscr{N}$ as soon as two objects among them belong to $\mathscr{N}$.

To a null system one associates a family of morphisms as follows. Define:

$$\mathscr{S} := \{f\colon X \to Y,\ \text{there exists a d.t. } X \to Y \to Z \to T(X) \text{ with } Z \in \mathscr{N}\}. \tag{6.4.1}$$

Lemma 6.4.2. *$\mathscr{S}$ is a right and left multiplicative system.*

Proof. By reversing the arrows, it is enough to prove that $\mathscr{S}$ is a right multiplicative system.

S1 is obvious.

S2 follows from the octahedral axiom TR5 (see (6.1.2)).

S3: There exists a d. t. $W \xrightarrow{h} X \to X' \xrightarrow{+1}$ with $W \in \mathscr{N}$. The morphism $h \circ f \colon W \to Y$ gives rise to a d. t. $W \to Y \to Z \xrightarrow{+1}$ and by TR4 there exists a morphism of triangles

$$\begin{tikzcd}
W \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & X' \arrow[r, "+1"] \arrow[d, dotted] & {} \\
W \arrow[r] & Y \arrow[r] & Z \arrow[r, "+1"] & {}
\end{tikzcd}$$

S4 By replacing f with $f-g$, it is enough to check that if there exists $s \in \mathscr{S} \colon W \to X$ such that $f \circ s = 0$ then there exists $t \in \mathscr{S} \colon Y \to Z$ such that $t \circ f = 0$. Consider the diagram

$$\begin{tikzcd}
X' \arrow[r, "s"] & X \arrow[r, "k"] \arrow[dr, "f"'] & Z \arrow[r, "+1"] \arrow[d, dotted, "h"] & {} \\
 & & Y \arrow[d, dotted, "t"] & \\
 & & Y' &
\end{tikzcd}$$

By Proposition 6.2.3 the sequence

$$\mathrm{Hom}\,(Z,Y) \xrightarrow{f} \mathrm{Hom}\,(X,Y) \to \mathrm{Hom}\,(Z,Y) \xrightarrow{k} \mathrm{Hom}\,(X',Y).$$

is exact. Since $f \circ s = 0$, the dotted arrow h may be completed, making the diagram commutative. Then we embed h in a d. t. and obtain the arrow t. Since $t \circ h = 0$, we get $t \circ f = 0$. Since $Z \in \mathscr{N}$, $t \in \mathscr{S}$. ☐

Theorem 6.4.3. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system in $\mathscr{D}$ and let $\mathscr{S}$ be as in* (6.4.1). *Then*

(i) *Denote as usual by $\mathscr{D}_{\mathscr{S}}$ the localization of $\mathscr{D}$ by $\mathscr{S}$ and by Q the localization functor. Then $\mathscr{D}_{\mathscr{S}}$ is an additive category endowed with an automorphism* (*the image of T, still denoted by T*)*.*

(ii) *Define a d.t. in $\mathscr{D}_{\mathscr{S}}$ as being isomorphic to the image by Q of a d.t. in $\mathscr{D}$. Then $\mathscr{D}_{\mathscr{S}}$ is a triangulated category.*

(iii) *If $X \in \mathscr{N}$, then $Q(X) \simeq 0$.*

(iv) *Let $F \colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories such that $F(X) \simeq 0$ for any $X \in \mathscr{N}$. Then F factors uniquely through Q.*

The proof being straightforward but tedious, it will not be given here. For a complete proof, see for example [KS06].

Notation 6.4.4. We will write $\mathscr{D}/\mathscr{N}$ instead of $\mathscr{D}_{\mathscr{S}}$.

Now consider a full triangulated subcategory $\mathscr{I}$ of $\mathscr{D}$. denote by $\mathscr{N} \cap \mathscr{I}$ the full subcategory of $\mathscr{D}$ whose objects are $\mathrm{Ob}(\mathscr{N}) \cap \mathrm{Ob}(\mathscr{I})$. This is clearly a null system in $\mathscr{I}$.

Proposition 6.4.5. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume condition* (i) *or* (ii) *below*

(i) *any morphism $Y \to Z$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Y \to Z' \to Z$ with $Z' \in \mathscr{N} \cap \mathscr{I}$,*

(ii) *any morphism $Z \to Y$ with $Y \in \mathscr{I}$ and $Z \in \mathscr{N}$ factorizes as $Z \to Z' \to Y$ with $Z' \in \mathscr{N} \cap \mathscr{I}$.*

Then the functor $\mathscr{I}/(\mathscr{N} \cap \mathscr{I}) \to \mathscr{D}/\mathscr{N}$ is fully faithful.

Proof. We shall apply Proposition 3.2.1. We may assume (ii), the case (i) being deduced by considering $\mathscr{D}^{\mathrm{op}}$. Let $f\colon Y \to X$ be a morphism in $\mathscr{S}$ with $Y \in \mathscr{I}$. We shall show that there exists $g\colon X \to W$ with $W \in \mathscr{I}$ and $g \circ f \in \mathscr{S}$. The morphism f is embedded in a d.t. $Y \to X \to Z \to T(Y)$ with $Z \in \mathscr{N}$. By the hypothesis, the morphism $Z \to T(Y)$ factorizes through an object $Z' \in \mathscr{N} \cap \mathscr{I}$. We may embed $Z' \to T(Y)$ into a d.t. and obtain a commutative diagram of d.t.:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "\mathrm{id}"] & X \arrow[r] \arrow[d, dotted, "g"] & Z \arrow[r] \arrow[d] & T(Y) \arrow[d, "\mathrm{id}"] \\
Y \arrow[r] & W \arrow[r] & Z' \arrow[r] & T(Y)
\end{tikzcd}$$

By TR4, the dotted arrow g may be completed and Z' belonging to $\mathscr{N}$, this implies that $g \circ f \in \mathscr{S}$. ☐

Proposition 6.4.6. *Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume conditions* (i) *or* (ii) *below:*

(i) *for any $X \in \mathscr{D}$, there exists a d.t. $X \to Y \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,*

(ii) *for any $X \in \mathscr{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$.*

Then $\mathscr{I}/\mathscr{N} \cap \mathscr{I} \to \mathscr{D}/\mathscr{N}$ is an equivalence of categories.

Proof. Apply Corollary 3.2.2. ☐

Localization of triangulated functors

Let $F\colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories and let $\mathscr{N}$ be a null system in $\mathscr{D}$. One defines the localization of F similarly as in the usual case, replacing all categories and functors by triangulated ones. Applying Proposition 3.3.2, we get:

Proposition 6.4.7. *Let $F\colon \mathscr{D} \to \mathscr{D}'$ be a functor of triangulated categories. Let $\mathscr{N}$ a null system of $\mathscr{D}$ and $\mathscr{I}$ a full triangulated subcategory of $\mathscr{D}$. Assume*

(a) *for any* $X \in \mathscr{D}$, *there exists a d.t.* $X \to Y \to Z \to T(X)$ *with* $Z \in \mathscr{N}$ *and* $Y \in \mathscr{I}$,

(b) *for any* $Y \in \mathscr{N} \cap \mathscr{I}$, $F(Y) \simeq 0$.

Then F *is right localizable.*

One defines $F_{\mathscr{N}}$ by the diagram:

$$\begin{tikzcd}
\mathscr{D} \arrow[rr] & & \mathscr{D}/\mathscr{N} \arrow[dd, dotted, "F_{\mathscr{N}}"] \\
\mathscr{I} \arrow[u] \arrow[r] \arrow[rrd] & \mathscr{I}/\mathscr{I}\cap\mathscr{N} \arrow[ru, "\sim"] \arrow[rd] & \\
& & \mathscr{D}'
\end{tikzcd}$$

If one replaces condition (a) in Proposition 6.4.7 by the condition

(a)' for any $X \in \mathscr{D}$, there exists a d.t. $Y \to X \to Z \to T(X)$ with $Z \in \mathscr{N}$ and $Y \in \mathscr{I}$,

one gets that F is left localizable.

Finally, let us consider triangulated bifunctors, i.e., bifunctors which are additive and triangulated with respect to each of their arguments.

Proposition 6.4.8. *Let* $\mathscr{D}, \mathscr{N}, \mathscr{I}$ *and* $\mathscr{D}', \mathscr{N}', \mathscr{I}'$ *be as in Proposition 6.4.7. Let* $F\colon \mathscr{D} \times \mathscr{D}' \to \mathscr{D}''$ *be a triangulated bifunctor. Assume:*

(a) *for any* $X \in \mathscr{D}$, *there exists a d.t.* $X \to Y \to Z \to T(X)$ *with* $Z \in \mathscr{N}$ *and* $Y \in \mathscr{I}$

(b) *for any* $X' \in \mathscr{D}'$, *there exists a d.t.* $X' \to Y' \to Z' \to T(X')$ *with* $Z' \in \mathscr{N}'$ *and* $Y' \in \mathscr{I}'$

(c) *for any* $Y \in \mathscr{I}$ *and* $Y' \in \mathscr{I}' \cap \mathscr{N}'$, $F(Y, Y') \simeq 0$,

(d) *for any* $Y \in \mathscr{I} \cap \mathscr{N}$ *and* $Y' \in \mathscr{I}'$, $F(Y, Y') \simeq 0$.

Then F *is right localizable.*

One denotes by $F_{\mathscr{N}\mathscr{N}'}$ its localization.

Of course, there exists a similar result for left localizable functors by reversing the arrows in the hypotheses (a) and (b) above.

Localization and direct sums

Proposition 6.4.9. *Let* $\mathscr{D}$ *be a triangulated category admitting small direct sums (hence, a small direct sum of d. t. is again a d. t.) and let* $\mathscr{N}$ *be a null system in* $\mathscr{D}$ *stable by such direct sums. Then* $\mathscr{D}/\mathscr{N}$ *admits small direct sums and the localization functor* Q *commutes with such direct sums.*

Proof. We shall follow [KS06, Prop. 10.2.8].

Let $\{X_i\}_{i\in I}$ be a small family of objects in $\mathscr{D}$ and let $Y \in \mathscr{D}$. A morphism $u\colon Q(\oplus_i X_i) \to Y$ defines for each i a morphism $u_i\colon Q(X_i) \to Y$. (As far as there is no risk of confusion, we write Y instead of $Q(Y)$.) Hence we have a natural map

$$\theta\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(\oplus_i X_i), Y) \xrightarrow{\sim} \prod_i \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Y).$$

In order to prove that $Q(\oplus_i X_i)$ is the direct sum of the family $Q(X_i)$, it is enough to check that θ is bijective for any $Y \in \mathscr{D}$.

(i) The map θ is surjective. Consider a family of morphisms $u_i\colon \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(Q(X_i), Y)$. We represent each u_i by a morphism $v_i\colon X'_i \to Y$ together with a d. t. $X'_i \to X_i \to Z_i \xrightarrow{+1}$ with $Z_i \in \mathscr{N}$. We get a morphism $v\colon \oplus_i X'_i \to Y$ and, for each i, a d. t. $\oplus_i X'_i \to \oplus_i X_i \to \oplus_i Z_i \xrightarrow{+1}$. By the hypothesis, $\oplus_i Z_i \in \mathscr{N}$ and it follows that v defines a morphism $Q(\oplus_i X'_i) \to Y$ in $\mathscr{D}/\mathscr{N}$.

(ii) The map θ is injective. Assume that the composition $Q(X_j) \to Q(\oplus_i X_i) \xrightarrow{u} Q(Y)$ is 0 for all $j \in I$. The morphism u may be represented by morphisms $\oplus_i X_i \xrightarrow{v} Y' \xleftarrow{s} Y$ with $s \in \mathscr{S}$ where $\mathscr{S}$ is the multiplicative system associated with $\mathscr{N}$ and the image of the composition $X_j \to \oplus_i X_i \xrightarrow{v} Y'$ is zero in $\mathscr{D}/\mathscr{N}$. By the result of Exercise 6.7 for each i there exists $Z_j \in \mathscr{N}$ such that this composition factorizes as $X_j \to Z_j \to Y'$. Therefore, $\oplus_j X_j \to Y'$ factorizes as $\oplus_j X_j \to \oplus_j Z_j \to Y'$ and thus $Q(u) = 0$. ☐

Exercises to Chapter 6

Exercise 6.1. Let $\mathscr{D}$ be a triangulated category and consider a commutative diagram in $\mathscr{D}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X),
\end{tikzcd}$$

Assume that $T(f) \circ h' = 0$ and the first row is a d.t. Prove that the second row is also a d.t. under one of the hypotheses:

(i) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(P, X) \to \mathrm{Hom}_{\mathscr{D}}(P, Y) \to \mathrm{Hom}_{\mathscr{D}}(P, Z') \to \mathrm{Hom}_{\mathscr{D}}(P, T(X)),$$

(ii) for any $P \in \mathscr{D}$, the sequence below is exact:

$$\mathrm{Hom}_{\mathscr{D}}(T(Y), P) \to \mathrm{Hom}_{\mathscr{D}}(T(X), P) \to \mathrm{Hom}_{\mathscr{D}}(Z', P) \to \mathrm{Hom}_{\mathscr{D}}(Y, P).$$

Exercise 6.2. Let $\mathscr{D}$ be a triangulated category and let $X_1 \to Y_1 \to Z_1 \to T(X_1)$ and $X_2 \to Y_2 \to Z_2 \to T(X_2)$ be two d.t. Show that $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to Z_1 \oplus Z_2 \to T(X_1) \oplus T(X_2)$ is a d.t.

In particular, $X \to X \oplus Y \to Y \xrightarrow{0} T(X)$ is a d.t.

(Hint: Consider a d.t. $X_1 \oplus X_2 \to Y_1 \oplus Y_2 \to H \to T(X_1) \oplus T(X_2)$ and construct the morphisms $H \to Z_1 \oplus Z_2$, then apply the result of Exercise 6.1.)

Exercise 6.3. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$ be a d.t. in a triangulated category.

(i) Prove that if $h = 0$, this d.t. is isomorphic to $X \to X \oplus Z \to Z \xrightarrow{0} T(X)$.

(ii) Prove the same result by assuming now that there exists $k : Y \to X$ with $k \circ f = \mathrm{id}_X$.

(Hint: to prove (i), construct the morphism $Y \to X \oplus Z$ by TR4, then use Proposition 6.2.4.)

Exercise 6.4. Let $X \xrightarrow{f} Y \to Z \to T(X)$ be a d.t. in a triangulated category. Prove that f is an isomorphism if and only if Z is isomorphic to 0.

Exercise 6.5. Let $f\colon X \to Y$ be a monomorphism in a triangulated category $\mathscr{D}$. Prove that there exist $Z \in \mathscr{D}$ and an isomorphism $h\colon Y \xrightarrow{\sim} X \oplus Z$ such that the composition $X \to Y \to X \oplus Z$ is the canonical morphism.

Exercise 6.6. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let Y be an object of $\mathscr{D}$ such that $\mathrm{Hom}_{\mathscr{D}}(Z, Y) \simeq 0$ for all $Z \in \mathscr{N}$. Prove that $\mathrm{Hom}_{\mathscr{D}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}/\mathscr{N}}(X, Y)$.

Exercise 6.7. Let $\mathscr{D}$ be a triangulated category, $\mathscr{N}$ a null system and let $Q\colon \mathscr{D} \to \mathscr{D}/\mathscr{N}$ be the canonical functor.

(i) Let $f\colon X \to Y$ be a morphism in $\mathscr{D}$ and assume that $Q(f) = 0$ in $\mathscr{D}/\mathscr{N}$. Prove that there exists $Z \in \mathscr{N}$ such that f factorizes as $X \to Z \to Y$.

(ii) For $X \in \mathscr{D}$, prove that $Q(X) \simeq 0$ if and only if there exists Y such that $X \oplus Y \in \mathscr{N}$ and this last condition is equivalent to $X \oplus TX \in \mathscr{N}$.

Chapter 7

Derived categories

This chapter is devoted to derived categories. Recall that the homotopy category $\mathrm{K}(\mathscr{C})$ of an additive category $\mathscr{C}$ is triangulated. When $\mathscr{C}$ is abelian, the cohomology functor $H^0\colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ is cohomological and the derived category $\mathrm{D}(\mathscr{C})$ of $\mathscr{C}$ is obtained by localizing $\mathrm{K}(\mathscr{C})$ with respect to the family of quasi-isomorphisms. We explain here this construction, with some examples. We also construct the right derived functor of a left exact functor as well as a bifunctor. Some classical examples are discussed.

Finally, we state, without proof, the Brown representability theorem, a fundamental result for applications.

The reader shall be aware that in general, the derived category $\mathrm{D}^+(\mathscr{C})$ of a $\mathscr{U}$-category $\mathscr{C}$ is no more a $\mathscr{U}$-category (see Remark 7.2.7).

Some references: [GM96, Har66, KS90, KS06, Ver96, Wei94].

7.1 Derived categories

Construction of the derived category

From now on, $\mathscr{C}$ will denote an abelian category.

Recall that if $f\colon X \to Y$ is a morphism in $\mathrm{C}(\mathscr{C})$, one says that f is a quasi-isomorphism (a qis, for short) if $H^k(f)\colon H^k(X) \to H^k(Y)$ is an isomorphism for all k. One extends this definition to morphisms in $\mathrm{K}(\mathscr{C})$.

If one embeds f into a d.t. $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$, then f is a qis iff $H^k(Z) \simeq 0$ for all $k \in \mathbb{Z}$, that is, if Z is qis to 0.

Proposition 7.1.1. *Let $\mathscr{C}$ be an abelian category. The functor $H^0\colon K(\mathscr{C}) \to \mathscr{C}$ is a cohomological functor.*

Proof. Let $X \xrightarrow{f} Y \to Z \xrightarrow{+1}$ be a d.t. Then it is isomorphic to $X \to Y \xrightarrow{\alpha(f)} \mathrm{Mc}(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{+1}$. Since the sequence in $\mathrm{C}(\mathscr{C})$:

$$0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0$$

is exact, it follows from Theorem 5.5.8 that the sequence

$$H^k(Y) \to H^k(\mathrm{Mc}(f)) \to H^{k+1}(X)$$

is exact. Therefore, $H^k(Y) \to H^k(Z) \to H^{k+1}(X)$ is exact. □

Corollary 7.1.2. *Let* $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ *be an exact sequence in* $\mathrm{C}(\mathscr{C})$ *and define* $\varphi\colon \mathrm{Mc}(f) \to Z$ *as* $\varphi^n = (0, g^n)$. *Then* φ *is a qis.*

Proof. Consider the exact sequence in $\mathrm{C}(\mathscr{C})$:

$$0 \to M(\mathrm{id}_X) \xrightarrow{\gamma} \mathrm{Mc}(f) \xrightarrow{\varphi} Z \to 0$$

where $\gamma^n\colon (X^{n+1} \oplus X^n) \to X^{n+1} \oplus Y^n$ is defined by: $\gamma^n = \begin{pmatrix} \mathrm{id}_{X^{n+1}} & 0 \\ 0 & f^n \end{pmatrix}$. Since $H^k(\mathrm{Mc}(\mathrm{id}_X)) \simeq 0$ for all k, we get the result by Proposition 7.1.1. ☐

We shall localize $\mathrm{K}(\mathscr{C})$ with respect to the family of objects qis to zero (see Section 6.4). Define:

$$N(\mathscr{C}) = \{X \in \mathrm{K}(\mathscr{C}); H^k(X) \simeq 0 \text{ for all } k\}.$$

One also defines $N^*(\mathscr{C}) = N(\mathscr{C}) \cap \mathrm{K}^*(\mathscr{C})$ for $^* = \mathrm{b}, +, -$.

Clearly, $N^*(\mathscr{C})$ is a null system in $\mathrm{K}^*(\mathscr{C})$.

Definition 7.1.3. One defines the derived categories $\mathrm{D}^*(\mathscr{C})$ as $\mathrm{K}^*(\mathscr{C})/N^*(\mathscr{C})$, where $* = \mathrm{ub}, \mathrm{b}, +, -$. One denotes by Q the localization functor $\mathrm{K}^*(\mathscr{C}) \to \mathrm{D}^*(\mathscr{C})$.

By Theorem 6.4.3, these are triangulated categories. Note that:

- a quasi-isomorphism in $\mathrm{K}(\mathscr{C})$ becomes an isomorphism in $\mathrm{D}(\mathscr{C})$,

Recall the truncation functors given in (5.5.9). These functors send a complex homotopic to zero to a complex homotopic to zero, hence are well defined on $\mathrm{K}^+(\mathscr{C})$. Moreover, they send a qis to a qis. Hence the functors below are well defined:

$$\begin{aligned} H^j(\,\bullet\,)&\colon \mathrm{D}(\mathscr{C}) \to \mathscr{C}, \\ \tau^{\leq n}, \widetilde{\tau}^{\leq n}&\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^-(\mathscr{C}), \\ \tau^{\geq n}, \widetilde{\tau}^{\geq n}&\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}). \end{aligned}$$

Note that:

- there are isomorphisms of functors
$$\tau^{\leq n} \simeq \widetilde{\tau}^{\leq n}, \quad \tau^{\geq n} \simeq \widetilde{\tau}^{\geq n},$$
- $H^j(\,\bullet\,)$ is a cohomological functor on $\mathrm{D}^*(\mathscr{C})$ (apply Proposition 7.1.1).

In particular, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{+1}$ is a d.t. in $\mathrm{D}(\mathscr{C})$, we get a long exact sequence:

$$\cdots \to H^k(X) \to H^k(Y) \to H^k(Z) \to H^{k+1}(X) \to \cdots \tag{7.1.1}$$

Let $X \in \mathrm{K}(\mathscr{C})$, with $H^j(X) = 0$ for $j > n$. Then the morphism $\tau^{\leq n}X \to X$ in $\mathrm{K}(\mathscr{C})$ is a qis, hence an isomorphism in $\mathrm{D}(\mathscr{C})$.

It follows from Proposition 6.4.5 that $\mathrm{D}^+(\mathscr{C})$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j << 0$, and similarly for $\mathrm{D}^-(\mathscr{C}), \mathrm{D}^{\mathrm{b}}(\mathscr{C})$. Moreover, $\mathscr{C}$ is equivalent to the full subcategory of $\mathrm{D}(\mathscr{C})$ consisting of objects X satisfying $H^j(X) \simeq 0$ for $j \neq 0$. For $a, b \in \mathbb{Z} \sqcup \{-\infty\} \sqcup \{+\infty\}$ with $a \leq b$, one sets

$$\mathrm{D}^{[a,b]}(\mathscr{C}) := \{X \in \mathrm{D}(\mathscr{C}); H^j(X) \simeq 0 \text{ for } j \notin [a,b]\}. \tag{7.1.2}$$

One defines similarly $\mathrm{D}^{\geq k}(\mathscr{C})$, $\mathrm{D}^{\leq k}(\mathscr{C})$, etc.

Definition 7.1.4. Let X, Y be objects of $\mathscr{C}$. One sets

$$\mathrm{Ext}^k_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y[k]).$$

Notation 7.1.5. Let A be a ring. We shall write for short $\mathrm{D}^*(A)$ instead of $\mathrm{D}^*(\mathrm{Mod}(A))$, for $* = \varnothing, \mathrm{b}, +, -$.

Remark 7.1.6. (i) Let $X \in \mathrm{K}(\mathscr{C})$ and let $Q(X)$ denote its image in $\mathrm{D}(\mathscr{C})$. One can prove that:

$$Q(X) \simeq 0 \Leftrightarrow X \text{ is qis to } 0 \text{ in } \mathrm{K}(\mathscr{C}).$$

(ii) Let $f\colon X \to Y$ be a morphism in $\mathrm{C}(\mathscr{C})$. Then $f \simeq 0$ in $\mathrm{D}(\mathscr{C})$ iff there exists X' and a qis $g\colon X' \to X$ such that $f \circ g$ is homotopic to 0, or else iff there exists Y' and a qis $h\colon Y \to Y'$ such that $h \circ f$ is homotopic to 0.

Remark 7.1.7. Consider an exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathscr{C}$. It gives rise to a d.t. $X \to Y \to Z \to X[1]$ in $\mathrm{D}(\mathscr{C})$. Consider the morphism $\gamma\colon Z \to X[1]$ in $\mathrm{D}(\mathscr{C})$. It defines morphisms $H^k(\gamma)\colon H^k(Z) \to H^{k+1}(X)$ is 0 for all $k \in \mathbb{Z}$ and X and Z being concentrated in degree 0, we get that $H^k(\gamma) \simeq 0$ for all $k \in \mathbb{Z}$. However, γ is *not* the zero morphism in $\mathrm{D}(\mathscr{C})$ in general (this happens if the short exact sequence splits). In fact, let us apply the cohomological functor $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, \bullet)$ to the d.t. above. It gives rise to the long exact sequence:

$$\cdots \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Z) \xrightarrow{\tilde{\gamma}} \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, X[1]) \to \cdots$$

where $\tilde{\gamma} = \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, \gamma)$. Since $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Y) \to \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(W, Z)$ is not an epimorphism in general, $\tilde{\gamma}$ is not zero. Therefore γ is not zero in general (see Theorem 7.4.9 below). The morphism γ may be described as follows (see Example 4.2.4), where φ is a qis in $\mathrm{C}(\mathscr{C})$:

$$\begin{tikzcd}
Z:= & 0 \arrow[r] & 0 \arrow[r] & Z \arrow[r] & 0 \\
\mathrm{Mc}(f) := \arrow[u, "\varphi"] \arrow[d, "\beta(f)"'] & 0 \arrow[r] & X \arrow[r, "f"] \arrow[u] \arrow[d, "\mathrm{id}"'] & Y \arrow[r] \arrow[u] \arrow[d] & 0 \\
X[1]:= & 0 \arrow[r] & X \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

By using the exact sequences (**??**), we get:

Proposition 7.1.8. *Let* $X \in \mathrm{D}(\mathscr{C})$.

(i) *There are d.t. in* $\mathrm{D}(\mathscr{C})$*:*

$$\begin{gathered}
\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \xrightarrow{+1}, \\
\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)[-n] \xrightarrow{+1}, \\
H^n(X)[-n] \to \tau^{\geq n}X \to \tau^{\geq n+1}X \xrightarrow{+1} .
\end{gathered} \tag{7.1.3}$$

(ii) *Moreover,* $H^n(X)[-n] \simeq \tau^{\leq n}\tau^{\geq n}X \simeq \tau^{\geq n}\tau^{\leq n}X$.

Corollary 7.1.9. *Let $\mathscr{C}$ be an abelian category and assume that for any $Y, Z \in \mathscr{C}$, $\mathrm{Ext}^k_{\mathscr{C}}(Y, Z) \simeq 0$ for $k \geq 2$. Let $X \in \mathrm{D^b}(\mathscr{C})$. Then:*

$$X \simeq \bigoplus_j H^j(X)\,[-j].$$

Proof. Call *amplitude* of X the smallest integer k such that $H^j(X) \simeq 0$ for j not belonging to some interval of length k. If $k = 0$, this means that there exists some i with $H^j(X) = 0$ for $j \neq i$, hence $X \simeq H^i(X)\,[-i]$. Now we argue by induction on the amplitude and we assume the result is proved for all X with amplitude $\leq n-1$. Let X with amplitude $\leq n$. Of course, we may assume (for simplicity in the notations) that X is concentrated in degree ≥ 0. Consider the d.t. (7.1.3):

$$\tau^{\leq n-1}X \to \tau^{\leq n}X \to H^n(X)\,[-n] \xrightarrow{+1} .$$

By the induction hypothesis, $\tau^{\leq n-1}X \simeq \oplus_{j<n} H^j(X)\,[-j]$. Now we have

$$\mathrm{Hom}_{\mathrm{D^b}(\mathscr{C})}(H^n(X)[-n], H^j(X)\,[-j+1]) \simeq \mathrm{Hom}_{\mathrm{D^b}(\mathscr{C})}(H^n(X), H^j(X)\,[n-j+1])$$

and these groups are 0 for $j < n$ by the hypothesis. Therefore,

$$\mathrm{Hom}_{\mathrm{D^b}(\mathscr{C})}(H^n(X)[-n], \tau^{\leq n-1}X\,[1]) \simeq 0$$

and the result follows from Exercise 6.3, ☐

Example 7.1.10. If a ring A is a principal ideal domain (such as a field, or $\mathbb{Z}$, or $\mathbf{k}[x]$ for $\mathbf{k}$ a field), then the category $\mathrm{Mod}(A)$ satisfies the hypotheses of Corollary 7.1.9.

7.2 Resolutions

Definition 7.2.1. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. We say that $\mathscr{J}$ is cogenerating if for all X in $\mathscr{C}$, there exist $Y \in \mathscr{J}$ and a monomorphism $X \rightarrowtail Y$.

If $\mathscr{J}$ is cogenerating in $\mathscr{C}^{\mathrm{op}}$, one says that $\mathscr{J}$ is generating in $\mathscr{C}$.

Theorem 7.2.2. *Assume $\mathscr{J}$ is cogenerating. Then for any $a \in \mathbb{Z}$ and $X^\bullet \in \mathrm{C}^{\geq a}(\mathscr{C})$, there exist $Y^\bullet \in \mathrm{C}^{\geq a}(\mathscr{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Proof. We shall follows the proof of [KS06, Lem. 13.2.1].
Let $X^\bullet \in \mathrm{C}^{\geq a}(\mathscr{C})$. We shall construct by induction on p a complex $Y^\bullet_{\leq p}$ in $\mathscr{J}$ and a morphism $f \colon X^\bullet \to Y^\bullet_{\leq p}$ such that $H^k(X^\bullet) \to H^k(Y^\bullet_{\leq p})$ is an isomorphism for $k < p$ and is a monomorphism for $k = p$, visualized by the diagram

$$\begin{tikzcd}
X^\bullet := & \cdots \arrow[r] & X^{p-1} \arrow[r, "d_X^{p-1}"] \arrow[d, "f^{p-1}"] & X^p \arrow[r, "d_X^p"] \arrow[d, "f^p"] & X^{p+1} \arrow[r, "d_X^{p+1}"] & \cdots \\
Y^\bullet_{\leq p} := & \cdots \arrow[r] & Y^{p-1} \arrow[r, "d_Y^{p-1}"] & Y^p & &
\end{tikzcd}$$

For $p < a$, choose $Y^\bullet_{\leq p} = 0$. Now assume that $Y^\bullet_{\leq p}$ has been constructed. Set

$$Z^p = \mathrm{Coker}\, d_Y^{p-1} \oplus_{\mathrm{Coker}\, d_X^{p-1}} \ker d_X^{p+1},$$
$$W^p = \mathrm{Coker}\, d_Y^{p-1} \oplus_{\mathrm{Coker}\, d_X^{p-1}} X^{p+1}.$$

(Recall that in an abelian category, given two morphisms $Z \to X$ and $Z \to Y$, $X \oplus_Z Y$ is the cokernel of $Z \to X \oplus Y$.) Hence, there is a monomorphism $Z^p \hookrightarrow W^p$. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^p(X^\bullet) \arrow[r] \arrow[d, "="] & \operatorname{Coker} d_X^{p-1} \arrow[r] \arrow[d] & \ker d_X^{p+1} \arrow[r] \arrow[d] & H^{p+1}(X^\bullet) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H^p(X^\bullet) \arrow[r] & \operatorname{Coker} d_Y^{p-1} \arrow[r] & Z^p \arrow[r] & H^{p+1}(X^\bullet) \arrow[r] & 0
\end{tikzcd} \tag{7.2.1}$$

The top row is clearly exact. The sequence $\operatorname{Coker} d_X^{p-1} \to \operatorname{Coker} d_Y^{p-1} \oplus \ker d_X^{p+1} \to H^{p+1}(X^\bullet) \to 0$ defines the morphism $Z^p \to H^{p+1}(X^\bullet)$ and one checks that the sequence $\operatorname{Coker} d_Y^{p-1} \to Z^p \to H^{p+1}(X^\bullet) \to 0$ is exact. Denote by K^p the kernel of $\operatorname{Coker} d_Y^{p-1} \to Z^p$. We get a morphism $u\colon H^p(X^\bullet) \to K^p$ which is a monomorphism by the induction hypothesis and which is an epimorphism thanks to the fact that the middle square in (7.2.1) is co-Cartesian (see [KS06, Exe. 8.21]). Therefore, Diagram 7.2.1 is exact. Since $\mathscr{J}$ is cogenerating, we may find a monomorphism $W^p \hookrightarrow Y^{p+1}$ with $Y^{p+1} \in \mathscr{J}$. The natural morphisms $X^{p+1} \to W^p$ and $Y^p \to W^p$ define the morphisms $f^{p+1}\colon X^{p+1} \to Y^{p+1}$ and $d_Y^p\colon Y^p \to Y^{p+1}$. Let $Y^\bullet_{\leq p+1}$ be the complex so constructed. Then

$$H^p(Y^\bullet_{\leq p+1}) \simeq \ker\bigl(\operatorname{Coker} d_Y^{p-1} \to Y^{p+1}\bigr) \simeq \ker\bigl(\operatorname{Coker} d_Y^{p-1} \to Z^p\bigr) \simeq H^p(X^\bullet).$$

Finally, the monomorphism $Z^p \to Y^{p+1}$ induces the monomorphism

$$\begin{aligned} H^{p+1}(X^\bullet) &\simeq \operatorname{Coker}\bigl(\operatorname{Coker} d_Y^{p-1} \to Z^p\bigr) \\ &\to \operatorname{Coker}\bigl(\operatorname{Coker} d_Y^{p-1} \to Y^{p+1}\bigr) \simeq H^{p+1}(Y^\bullet_{\leq p+1}). \end{aligned}$$

□

We shall also have to consider the following situation. Consider the hypothesis

$$\begin{cases} \text{there exists } d \in \mathbb{N}_{>0} \text{ such that for any exact sequence} \\ Y_1 \to \cdots Y_d \to Y \to 0, \text{ with } Y_j \in \mathscr{J},\ 1 \leq j \leq d, \text{ we have } Y \in \mathscr{J}. \end{cases} \tag{7.2.2}$$

Corollary 7.2.3. *Assume $\mathscr{J}$ is cogenerating and satisfies (7.2.2). Then for any $X^\bullet \in \mathrm{C}^{[a,b]}(\mathscr{C})$, there exist $Y^\bullet \in \mathrm{C}^{[a,b+d+1]}(\mathscr{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.*

Proof. Let $X^\bullet \to Y^\bullet$ be a quasi-isomorphism given by Theorem 7.2.2, with $Y^\bullet \in \mathrm{C}^{\geq a}(\mathscr{J})$. Consider the truncation functor $\tau^{\leq j}$ of (5.5.9). It induces an isomorphism $\tau^{\leq j}(X^\bullet) \xrightarrow{\sim} X^\bullet$ for $j > b$ and a quasi-isomorphism for all j:

$$\tau^{\leq j}(X^\bullet) \xrightarrow{qis} \tau^{\leq j}(Y^\bullet).$$

Moreover, the sequence $Y^{b+1} \to Y^{b+2} \to \cdots$ is exact. Thanks to the hypothesis, we get that $\tau^{\leq b+d}(Y^\bullet)$ belongs to $\mathrm{C}^{[a,b+d+1]}(\mathscr{J})$ and this complex is qis to $X^\bullet$. □

In the sequel, for $\mathscr{J}$ an additive subcategory of $\mathscr{C}$, we set

$$N^+(\mathscr{J}) := N(\mathscr{C}) \cap \mathrm{K}^+(\mathscr{J}). \tag{7.2.3}$$

It is clear that $N^+(\mathscr{J})$ is a null system in $\mathrm{K}^+(\mathscr{J})$.

Applying Proposition 6.4.6, we get:

Corollary 7.2.4. *Let $\mathscr{J}$ be an additive subcategory of $\mathscr{C}$ and assume that $\mathscr{J}$ is cogenerating. Then*

(a) *For any $X^\bullet \in \mathrm{K}^+(\mathscr{C})$, there exists $Y^\bullet \in \mathrm{K}^+(\mathscr{J})$ and a qis $X^\bullet \to Y^\bullet$. Moreover, the natural functor $\theta\colon \mathrm{K}^+(\mathscr{J})/N^+(\mathscr{J}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence of categories.*

(b) *Assume moreover that $\mathscr{J}$ satisfies (7.2.2) and $X^\bullet \in \mathrm{K}^\mathrm{b}(\mathscr{C})$. Then we may choose $Y^\bullet \in \mathrm{K}^\mathrm{b}(\mathscr{J})$ and the natural functor $\theta\colon \mathrm{K}^\mathrm{b}(\mathscr{J})/N^\mathrm{b}(\mathscr{J}) \to \mathrm{D}^\mathrm{b}(\mathscr{C})$ is an equivalence of categories.*

Injective resolutions

In this subsection, $\mathscr{C}$ denotes an abelian category and $\mathscr{I}_{\mathscr{C}}$ its full additive subcategory consisting of injective objects. We shall asume

(7.2.4) the abelian category $\mathscr{C}$ admits enough injectives.

In other words, the category $\mathscr{I}_{\mathscr{C}}$ is cogenerating.

Proposition 7.2.5. (i) *Let $f^\bullet : X^\bullet \to I^\bullet$ be a morphism in $\mathrm{C}^+(\mathscr{C})$. Assume $I^\bullet$ belongs to $\mathscr{C}^+(\mathscr{I}_{\mathscr{C}})$ and $X^\bullet$ is exact. Then $f^\bullet$ is homotopic to 0.*

(ii) *Let $I^\bullet \in \mathrm{C}^+(\mathscr{I}_{\mathscr{C}})$ and assume $I^\bullet$ is exact. Then $I^\bullet$ is homotopic to 0.*

Proof. (i) Consider the solid diagram:

$$\begin{tikzcd}
X^{k-2} \arrow[r] & X^{k-1} \arrow[r] \arrow[d, "f^{k-1}"] \arrow[dl, dotted, "s^{k-1}"'] & X^k \arrow[r] \arrow[d, "f^k"] \arrow[dl, dotted, "s^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
I^{k-2} \arrow[r] & I^{k-1} \arrow[r] & I^k \arrow[r] & I^{k+1}
\end{tikzcd}$$

We shall construct by induction morphisms s^k satisfying:

$$f^k = s^{k+1} \circ d_X^k + d_I^{k-1} \circ s^k.$$

For $j << 0$, $s^j = 0$. Assume we have constructed the s^j for $j \leq k$. Define $g^k = f^k - d_I^{k-1} \circ s^k$. One has

$$\begin{aligned}
g^k \circ d_X^{k-1} &= f^k \circ d_X^{k-1} - d_I^{k-1} \circ s^k \circ d_X^{k-1} \\
&= f^k \circ d_X^{k-1} - d_I^{k-1} \circ f^{k-1} + d_I^{k-1} \circ d_I^{k-2} \circ s^{k-1} \\
&= 0.
\end{aligned}$$

Hence, g^k factorizes through $X^k/\operatorname{Im} d_X^{k-1}$. Since the complex $X^\bullet$ is exact, the sequence $0 \to X^k/\operatorname{Im} d_X^{k-1} \to X^{k+1}$ is exact. Consider

$$\begin{tikzcd}
0 \arrow[r] & X^k/\operatorname{Im} d_X^{k-1} \arrow[r] \arrow[d, "g^k"'] & X^{k+1} \arrow[dl, dotted, "s^{k+1}"'] \\
& I^k &
\end{tikzcd}$$

The dotted arrow may be completed by Proposition 5.3.2.
(ii) Apply the result of (i) with $X^\bullet = I^\bullet$ and $f = \mathrm{id}_X$. ☐

Corollary 7.2.6. *Assume that $\mathscr{C}$ admits enough injectives. Then* $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}}) \to \mathrm{D}^+(\mathscr{C})$ *is an equivalence of categories.*

Proof. According to Notation 7.2.3, $N^+(\mathscr{I}_{\mathscr{C}})$ is the subcategory of $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})$ consisting of complexes qis to 0. By Corollary 7.2.4, the natural functor $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})/N^+(\mathscr{I}_{\mathscr{C}}) \to \mathrm{D}^+(\mathscr{C})$ is an equivalence. To conclude, remark that the objects of $N^+(\mathscr{I}_{\mathscr{C}})$ are isomorphic to 0 in $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})$. Hence, $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})/N^+(\mathscr{I}_{\mathscr{C}})$ is equivalent to $\mathrm{K}^+(\mathscr{I}_{\mathscr{C}})$. □

Remark 7.2.7. Assume that $\mathscr{C}$ admits enough injectives. Then $\mathrm{D}^+(\mathscr{C})$ is a $\mathscr{U}$-category.

7.3 Derived functors

In this section, $\mathscr{C}$ and $\mathscr{C}'$ will denote abelian categories and $F\colon \mathscr{C} \to \mathscr{C}'$ a left exact functor. We shall construct the right derived functor $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ under suitable conditions, and in particular the j-th derived functor $R^jF\colon \mathscr{C} \to \mathscr{C}'$. Note that we do not assume that $\mathscr{C}$ admits enough injectives.

The functor F defines naturally a functor

$$\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{K}^+(\mathscr{C}').$$

For short, one often writes F instead of K^+F. Applying the results of § 6.4, we shall construct (under suitable hypotheses) the right localization of F. Recall Definition 7.3.3. By Lemma 7.3.4, $\mathrm{K}^+(F)$ sends $N^+(\mathscr{J})$ to $N^+(\mathscr{C}')$.

Definition 7.3.1. (a) If the functor $\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ admits a right localization (with respect to the qis in $\mathrm{K}^+(\mathscr{C})$), one says that F admits a right derived functor and one denotes by $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$ the right localization of F.

(b) If F admits a right derived functor, one sets for $X \in \mathscr{C}$, $R^jF(X) = H^j(RF(X))$.

(c) An object $X \in \mathscr{C}$ satisfying $R^jF(X) \simeq 0$ for all $j \neq 0$ is called F-acyclic.

(d) Assume that F admits a right derived functor. One says that F has cohomological dimension $\leq d$ with $d \in \mathbb{N}$ if for any $X \in \mathscr{C}$, $R^jF(X) \simeq 0$ for $j > d$. If such an integer d exists, one says that F has finite cohomological dimension.

There is a similar definition for right exact functor. The exact formulation is left to the reader.

Note that if $F\colon \mathscr{C} \to \mathscr{C}'$ is exact, it admits a right derived functor as well as a left derived functor and both coincide. In this case, one still denotes by F its localization.

Recall that if RF exists, then it sends distinguished triangles in $\mathrm{D}^+(\mathscr{C})$ to distinguished triangles in $\mathrm{D}^+(\mathscr{C}')$. In particular, we get:

Proposition 7.3.2. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ a left exact functor as above and let $0 \to X' \xrightarrow{f} X \xrightarrow{g} X'' \to 0$ be an exact sequence in $\mathscr{C}$. Then Then there exists a long exact sequence:*

$$0 \to F(X') \to F(X) \to \cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to \cdots .$$

Definition 7.3.3. Let $\mathscr{J}$ be a full additive subcategory of $\mathscr{C}$. One says that $\mathscr{J}$ is F-injective if:

(i) $\mathscr{J}$ is cogenerating,

(ii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ with $X' \in \mathscr{J}, X \in \mathscr{J}$, then $X'' \in \mathscr{J}$,

(iii) for any exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ with $X' \in \mathscr{J}$, the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact.

By considering $\mathscr{C}^{\mathrm{op}}$, one obtains the notion of an F-projective subcategory, F being right exact.

Lemma 7.3.4. *Assume $\mathscr{J}$ is F-injective and let $X^\bullet \in \mathrm{C}^+(\mathscr{J})$ be a complex qis to zero* (i.e. *$X^\bullet$ is exact*). *Then $F(X^\bullet)$ is qis to zero.*

Proof. We decompose $X^\bullet$ into short exact sequences (assuming that this complex starts at step 0 for convenience):

$$\begin{aligned}
&0 \to X^0 \to X^1 \to Z^1 \to 0\\
&0 \to Z^1 \to X^2 \to Z^2 \to 0\\
&\cdots\\
&0 \to Z^{n-1} \to X^n \to Z^n \to 0
\end{aligned}$$

By induction we find that all the Z^j's belong to $\mathscr{J}$, hence all the sequences:

$$0 \to F(Z^{n-1}) \to F(X^n) \to F(Z^n) \to 0$$

are exact. Hence the sequence

$$0 \to F(X^0) \to F(X^1) \to \cdots$$

is exact. □

Theorem 7.3.5. *Let $F\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and let $\mathscr{J} \subset \mathscr{C}$ be a full additive subcategory. Assume that $\mathscr{J}$ is F-injective. Then*

(a) *F admits a right derived functor $RF\colon \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathscr{C}')$.*

(b) *If moreover $\mathscr{J}$ satisfies* (7.2.2)*, then RF induces a functor $\mathrm{D}^{\mathrm{b}}(\mathscr{C}) \to \mathrm{D}^{\mathrm{b}}(\mathscr{C}')$.*

By reversing the arrows, one obtains a similar result for right exact functors.

Proof. (i) We shall apply Proposition 6.4.7. Condition (a) is satisfied thanks to Corollary 7.2.4. Condition (b) is satisfied thanks to Lemma 7.3.4.

(ii) The proof of (b) is similar. □

The construction of RF is vizualised by the diagram

$$\begin{tikzcd}
\mathrm{K}^+(\mathscr{J}) \arrow[r, "\mathrm{K}^+F"] \arrow[d, "Q"'] & \mathrm{K}^+(\mathscr{C}') \arrow[dd, "Q"] \\
\mathrm{K}^+(\mathscr{J})/N^+(\mathscr{J}) \arrow[d, "\sim"'] \arrow[dr] & \\
\mathrm{D}^+(\mathscr{C}) \arrow[r, dotted, "RF"'] & \mathrm{D}^+(\mathscr{C}').
\end{tikzcd}$$

Recall that the derived functor RF is triangulated, and does not depend on the category $\mathscr{J}$. Hence, if $X' \to X \to X'' \xrightarrow{+1}$ is a d.t. in $\mathrm{D}^+(\mathscr{C})$, then $RF(X') \to RF(X) \to RF(X'') \xrightarrow{+1}$ is a d.t. in $\mathrm{D}^+(\mathscr{C}')$. (Recall that an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathscr{C}$ gives rise to a d.t. in $\mathrm{D}(\mathscr{C})$.) Applying the cohomological functor H^0, we get the long exact sequence in $\mathscr{C}'$:

$$\cdots \to R^kF(X') \to R^kF(X) \to R^kF(X'') \to R^{k+1}F(X') \to \cdots$$

By considering the category $\mathscr{C}^{\mathrm{op}}$, one defines the notion of left derived functor of a right exact functor F.

Remark 7.3.6. Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor and assume that F admits a right derived functor.

(i) If the category $\mathscr{I}_{\mathscr{C}}$ of injective objects of $\mathscr{C}$ is cogenerating, then it is F-injective.

(ii) Denote by $\mathscr{I}_F$ the full additive subcategory of $\mathscr{C}$ consisting of F-acyclic objects and assume that this category if cogenerating. Then $\mathscr{I}_F$ is F-injective. Indeed, conditions (ii) and (iii) of Definition 7.3.3 are satisfied thanks to Proposition 7.3.2.

The next corollary, whose proof follows immediately from the construction of RF, gives an explicit construction of the derived functor.

Corollary 7.3.7. *Assume $\mathscr{J}$ is F-injective. Let $X \in \mathscr{C}$ and let $0 \to X \to Y^\bullet$ be a resolution of X with $Y^\bullet \in \mathrm{C}^+(\mathscr{J})$. Then for each n, there is an isomorphism $R^nF(X) \simeq H^n(F(Y^\bullet))$.*

In other words, in order to calculate the derived functors $R^jF(X)$ for $X \in \mathscr{C}$, it is enough to replace X with a right $\mathscr{J}$-resolution, apply F to this complex and take the j-th cohomology. This construction applies in particular if $\mathscr{C}$ has enough injectives.

Derived functor of a composition

Let $F \colon \mathscr{C} \to \mathscr{C}'$ and $G \colon \mathscr{C}' \to \mathscr{C}''$ be left exact functors of abelian categories. Then $G \circ F \colon \mathscr{C} \to \mathscr{C}''$ is left exact. Using the universal property of the localization, one shows that if F, G and $G \circ F$ are right derivable, then there exists a natural morphism of functors

$$R(G \circ F) \to RG \circ RF. \tag{7.3.1}$$

Theorem 7.3.8. *Assume that there exist full additive subcategories $\mathscr{J} \subset \mathscr{C}$ and $\mathscr{J}' \subset \mathscr{C}'$ such that $\mathscr{J}$ is F-injective, $\mathscr{J}'$ is G-injective and $F(\mathscr{J}) \subset \mathscr{J}'$. Then $\mathscr{J}$ is $(G \circ F)$-injective and the morphism in (7.3.1) is an isomorphism: $R(G \circ F) \xrightarrow{\sim} RG \circ RF$.*

Proof. (i) The fact that $\mathscr{J}$ is $(G \circ F)$ injective follows immediately from the definition.

(ii) Let $X \in \mathrm{K}^+(\mathscr{C})$ and $Y \in \mathrm{K}^+(\mathscr{J})$ together with a qis $X \to Y$. Then $RF(X)$ is represented by the complex $F(Y)$ which belongs to $\mathrm{K}^+(\mathscr{J}')$. Hence $RG(RF(X))$ is represented by $G(F(Y)) = (G \circ F)(Y)$, and this last complex also represents $R(G \circ F)(Y)$ since $Y \in \mathrm{C}^+(\mathscr{J})$ and $\mathscr{J}$ is $G \circ F$ injective. ☐

Note that in general F does not send injective objects of $\mathscr{C}$ to injective objects of $\mathscr{C}'$. That is why the notion of an “F-injective” category is important.

Corollary 7.3.9. *Assume that there exists a full additive subcategory $\mathscr{J} \subset \mathscr{C}$ such that $\mathscr{J}$ is F-injective and assume that G is exact. Then $\mathscr{J}$ is $(G\circ F)$-injective and the morphism in (7.3.1) is an isomorphism.*

Proof. If G is exact, then $\mathscr{C}'$ is G-injective. □

Corollary 7.3.10. *In the situation of Theorem 7.3.8, let $X \in \mathscr{C}$ and assume that $R^jF(X) \simeq 0$ for $j > 0$ and that F sends the objects of $\mathscr{J}$ to G-acyclic objects of $\mathscr{C}'$. Then $R^j(G\circ F)(X) \simeq (R^jG)(F(X))$.*

Proof. Let $X \to I^\bullet_X$ be a quasi-isomorphism with $I^\bullet_X \in \mathrm{C}^+(\mathscr{J})$. Denote by $\mathscr{I}_G$ the full additive subcategory of $\mathscr{C}'$ consisting of G-acyclic objects (see Remark 7.3.6). By the hypothesis, $F(I^\bullet_X)$ is qis to $F(X)$ and belongs to $\mathrm{C}^+(\mathscr{I}_G)$. Hence $R^j(G\circ F)(X)) \simeq H^j(G(F(I^\bullet_X)))$ by Corollary 7.3.7. Finally, $H^j(G(F(I^\bullet_X))) \simeq R^jGF(X)$. □

7.4 Bifunctors

Now consider three abelian categories $\mathscr{C}, \mathscr{C}', \mathscr{C}''$ and an *additive* bifunctor:

$$F\colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''.$$

We shall assume that F is left exact with respect to each of its arguments.

Let $X \in \mathrm{K}^+(\mathscr{C}), X' \in \mathrm{K}^+(\mathscr{C}')$ and assume that X or X' is homotopic to 0. Then one checks easily that $\mathrm{tot}(F(X,X'))$ is homotopic to zero. Hence one can naturally define:

$$\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \times \mathrm{K}^+(\mathscr{C}') \to \mathrm{K}^+(\mathscr{C}'')$$

Explain why

by setting:

$$\mathrm{K}^+F(X,X') = \mathrm{tot}(F(X,X')).$$

If there is no risk of confusion, we shall sometimes write F instead of K^+F.

Definition 7.4.1. If the functor $\mathrm{K}^+F\colon \mathrm{K}^+(\mathscr{C}) \times \mathrm{K}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}'')$ admits a right localization (with respect to the qis in $\mathrm{K}^+(\mathscr{C})$ and $\mathrm{K}^+(\mathscr{C}')$), one says that F admits a right derived functor and one denotes by $RF\colon \mathrm{D}^+(\mathscr{C}) \times \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C}'')$ the right localization of F.

One defines similarly the notion of left derived functor for a right exact bifunctor.

Definition 7.4.2. Let $\mathscr{J}$ and $\mathscr{J}'$ be additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. One says $(\mathscr{J}, \mathscr{J}')$ is F-injective if:

(i) for all $X' \in \mathscr{J}'$, $\mathscr{J}$ is $F(\,\bullet\,, X')$-injective,

(ii) for all $X \in \mathscr{J}$, $\mathscr{J}'$ is $F(X, \bullet\,)$-injective.

One defines similarly the notion of being G-projective for a right exact bifunctor G.

Theorem 7.4.3. *Let $F\colon \mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ and $\mathscr{J}'$ be additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. Assume that $(\mathscr{J},\mathscr{J}')$ is F-injective. Then F admits a right derived functor $RF\colon \mathrm{D}^+(\mathscr{C})\times\mathrm{D}^+(\mathscr{C}')\to\mathrm{D}^+(\mathscr{C}'')$.*

Proof. Let $X\in\mathrm{K}^+(\mathscr{J})$ and $X'\in\mathrm{K}^+(\mathscr{J}')$. If X or X' is qis to 0, then $F(X,X')$ is qis to zero. Indeed, all rows and column of $F(X,X')$ are exact.

To conclude, apply Proposition 6.4.8. □

There is a similar statement for a right exact bifunctor G, replacing F-injective with G-projective.

One naturally extends Definition 7.3.1 to bifunctors. The exact statement is left to the reader.

Proposition 7.4.4. *Let $F\colon \mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ and $\mathscr{J}'$ be two additive cogeneratimg subcategories of $\mathscr{C}$ and $\mathscr{C}'$ respectively. Assume*

(a) *for each $I\in\mathscr{J}$, the functor $F(I,\bullet)\colon\mathscr{C}'\to\mathscr{C}''$ is exact,*

(b) *for each $I'\in\mathscr{J}'$, the functor $F(\bullet,I')\colon\mathscr{C}\to\mathscr{C}''$ is exact.*

Then for all $Y\in\mathscr{C}'$, $\mathscr{J}$ is $F(\bullet,Y)$-injective and for all $X\in\mathscr{C}$, $\mathscr{J}'$ is $F(X,\bullet)$-injective.

Proof. (i) Let $0\to X'\to X\to X''\to 0$ be an exact sequence in $\mathscr{C}$. It is easily checked that if both functors $F(X',\bullet)$ and $F(X,\bullet)$ are exact, then so is $F(X'',\bullet)$.
(ii) Now consider the exact sequence as above and assume that $X'\in\mathscr{J}$. Let $Y\in\mathscr{C}'$ and let us prove that the sequence $0\to F(X',Y)\to F(X,Y)\to F(X'',Y)\to 0$ is exact. Consider a resolution $0\to Y\to Y^\bullet$ with the Y^j's in $\mathscr{J}'$ and consider the double complex

$$0\to F(X',Y^\bullet)\to F(X,Y^\bullet)\to F(X'',Y^\bullet)\to 0$$

The rows of this complex are exact. Hence, we get a long exact sequence (see Theorem 5.5.8)

$$0\to F(X',Y)\to F(X,Y)\to F(X'',Y)\to H^1(F(X',Y^\bullet))\to\cdots.$$

Since $F(X',\bullet)$ is exact and $H^1(Y^\bullet)\simeq 0$, $H^1(F(X',Y^\bullet))\simeq 0$. □

Lemma 7.4.5. *Let $F\colon \mathscr{C}\times\mathscr{C}'\to\mathscr{C}''$ and $\mathscr{J}$ $\mathscr{J}'$ be as in Proposition 7.4.4. Let $0\to X\to I_X^\bullet$ and $0\to Y\to I_Y^\bullet$ be resolutions of X and Y in $\mathscr{J}$ and $\mathscr{J}'$, respectively.*

Then, for $j\in\mathbb{Z}$, there is an isomorphism $H^jF(X,I_Y^\bullet)\simeq H^jF(I_X^\bullet,Y)$.

Proof. Consider the double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & F(I^0_X, Y) \arrow[r] \arrow[d] & F(I^1_X, Y) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F(X, I^0_Y) \arrow[r] \arrow[d] & F(I^0_X, I^0_Y) \arrow[r] \arrow[d] & F(I^1_X, I^0_Y) \arrow[r] \arrow[d] & {} \\
0 \arrow[r] & F(X, I^1_Y) \arrow[r] \arrow[d] & F(I^0_X, I^1_Y) \arrow[r] \arrow[d] & F(I^1_X, I^1_Y) \arrow[r] \arrow[d] & {} \\
 & {} & {} & {} &
\end{tikzcd}$$

The cohomology of the first row (resp. column) calculates $H^kF(I^\bullet_X, Y)$ (resp. $H^kF(X, I^\bullet_Y)$). Since the other rows and columns are exact by the hypotheses, the result follows from Theorem 5.6.4. ☐

Corollary 7.4.6. *Let $F\colon \mathscr{C} \times \mathscr{C}' \to \mathscr{C}''$ be a left exact bifunctor of abelian categories and let $\mathscr{J}$ and $\mathscr{J}'$ be cogeneratimg additive subcategories of $\mathscr{C}$ and $\mathscr{C}'$, respectively. Assume moreover that for all $X \in \mathscr{J}$, $F(X, \bullet)$ is exact and for all $X' \in \mathscr{J}'$, $F(\bullet, X')$ is exact. Then for all $X' \in \mathscr{C}'$, $\mathscr{J}$ is $F(\bullet, X')$-injective and for all $X \in \mathscr{C}$, $\mathscr{J}'$ is $F(X, \bullet)$-injective. Moreover, F admits a right derived functor, and for all $X \in \mathscr{C}$, $Y \in \mathscr{C}'$, one has:*

$$RF(X, Y) \simeq (RF(X, \bullet))(Y) \simeq (RF(\bullet, Y))(X). \tag{7.4.1}$$

Proof. The first part of the statement is nothing but Proposition 7.4.4. The fact that F admits a right derived functor follows from Theorem 7.4.3 and (7.4.1) follows from Lemma 7.4.5, ☐

Example 7.4.7. Assume $\mathscr{C}$ has enough injectives. Then

$$\mathrm{RHom}_{\mathscr{C}}\colon \mathrm{D}^-(\mathscr{C})^{\mathrm{op}} \times \mathrm{D}^+(\mathscr{C}) \to \mathrm{D}^+(\mathbb{Z})$$

exists and may be calculated as follows. Let $X \in \mathrm{D}^-(\mathscr{C})$ and $Y \in \mathrm{D}^+(\mathscr{C})$. There exists a qis in $\mathrm{K}^+(\mathscr{C})$, $Y \to I$, the I^j's being injective. Then:

$$\mathrm{RHom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}^\bullet_{\mathscr{C}}(X, I).$$

If $\mathscr{C}$ has enough projectives, and $P \to X$ is a qis in $\mathrm{K}^-(\mathscr{C})$, the P^j's being projective, one also has:

$$\mathrm{RHom}_{\mathscr{C}}(X, Y) \simeq \mathrm{Hom}^\bullet_{\mathscr{C}}(P, Y).$$

These isomorphisms hold in $\mathrm{D}^+(\mathbb{Z})$.

Example 7.4.8. Let A be a $\mathbf{k}$-algebra. By choosing the category of projective modules for $\mathscr{J}$ and $\mathscr{J}'$ in Theorem 7.4.3, we get that the bifunctor

$$\bullet \overset{\mathrm{L}}{\otimes}_A \bullet \colon \mathrm{D}^-(A^{\mathrm{op}}) \times \mathrm{D}^-(A) \to \mathrm{D}^-(\mathbf{k})$$

is well defined. Moreover,

$$N \overset{\mathrm{L}}{\otimes}_A M \simeq s(N \otimes_A P) \simeq s(Q \otimes_A M)$$

where P (resp. Q) is a complex of projective A-modules qis to M (resp. to N).

Note that instead of choosing the category of projective modules, we could have chosen that of flat modules. When working with sheaves, there are not enough projective modules in general, although they are enough flat modules.

In the preceding situation, one has:

$$\mathrm{Tor}_A^{-k}(N,M) \simeq H^k(N\overset{\mathrm{L}}{\otimes}_A M).$$

The functors $\mathrm{RHom}_{\mathscr{C}}$ and $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}$

Theorem 7.4.9. *Let $\mathscr{C}$ be an abelian category with enough injectives. Then for $X \in \mathrm{D}^-(\mathscr{C})$, $Y \in \mathrm{D}^+(\mathscr{C})$ and $j \in \mathbb{Z}$:*

$$H^j\mathrm{RHom}_{\mathscr{C}}(X,Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y\,[j]).$$

Proof. By Proposition 7.2.2, there exists $I_Y \in \mathrm{C}^+(\mathscr{I})$ and a qis $Y \to I_Y$. Then we have the isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X,Y[j]) &\simeq \mathrm{Hom}_{K(\mathscr{C})}(X, I_Y[j])\\ &\simeq H^0(\mathrm{Hom}^\bullet_{\mathscr{C}}(X, I_Y[j]))\\ &\simeq R^j\mathrm{Hom}_{\mathscr{C}}(X,Y),\end{aligned}$$

where the second isomorphism follows from Proposition 4.4.5. □

Recall that one has set

$$\mathrm{Ext}^j_{\mathscr{C}}(X,Y) := H^j\mathrm{RHom}_{\mathscr{C}}(X,Y).$$

Example 7.4.10. Let W be the Weyl algebra in one variable over a field $\mathbf{k}$ of characteristic 0: $W = \mathbf{k}[x,\partial]$ with the relation $[x,\partial] = -1$.

Let $\mathscr{O} = W/W\cdot\partial$, $\Omega = W/\partial\cdot W$ and let us calculate $\Omega\overset{\mathrm{L}}{\otimes}_W\mathscr{O}$. We have an exact sequence: $0 \to W \xrightarrow{\partial\cdot} W \to \Omega \to 0$. Therefore, Ω is qis to the complex

$$0 \to W^{-1} \xrightarrow{\partial\cdot} W^0 \to 0$$

where $W^{-1} = W^0 = W$ and W^0 is in degree 0. Then $\Omega\overset{\mathrm{L}}{\otimes}_W\mathscr{O}$ is qis to the complex

$$0 \to \mathscr{O}^{-1} \xrightarrow{\partial\cdot} \mathscr{O}^0 \to 0,$$

where $\mathscr{O}^{-1} = \mathscr{O}^0 = \mathscr{O}$ and $\mathscr{O}^0$ is in degree 0. Since $\partial\colon \mathscr{O} \to \mathscr{O}$ is surjective and has $\mathbf{k}$ as kernel, we obtain:

$$\Omega\overset{\mathrm{L}}{\otimes}_W\mathscr{O} \simeq \mathbf{k}[1].$$

Example 7.4.11. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1,\dots,x_n]$. This is a commutative Noetherian ring and it is known (Hilbert) that any finitely generated A-module M admits a finite free presentation of length at most n, *i.e.,* M is qis to a complex:

$$L := \quad 0 \to L^{-n} \to \cdots \xrightarrow{P_0} L^0 \to 0$$

where the L^j's are free of finite rank. Consider the left exact functor

$$\mathrm{Hom}_A(\,\bullet\,,A)\colon \mathrm{Mod}(A)^{\mathrm{op}} \to \mathrm{Mod}(A)$$

and denote for short by * its derived functor:

$$^* := \mathrm{RHom}_A(\,\bullet\,,A). \tag{7.4.2}$$

Since free A-modules are projective, we find that $\mathrm{RHom}_A(M,A)$ is isomorphic in $\mathrm{D^b}(A)$ to the complex

$$L^* := \quad 0 \leftarrow L^{-n*} \leftarrow \cdots \xleftarrow{P_0} L^{0*} \leftarrow 0.$$

Using (7.3.1), we find a natural morphism of functors

$$\mathrm{id} \to {}^{**} := {}^* \circ {}^*.$$

Applying * to the object $\mathrm{RHom}_A(M,A)$ we find:

$$\begin{aligned}\mathrm{RHom}_A(\mathrm{RHom}_A(M,A),A) &\simeq \mathrm{RHom}_A(L^*,A)\\ &\simeq L \simeq M.\end{aligned}$$

In other words, we have proved the isomorphism $M \simeq M^{**}$ in $\mathrm{D^b}(A)$.

Assume now $n = 1$, *i.e.*, $A = \mathbf{k}[x]$ and consider the natural morphism in $\mathrm{Mod}(A)$: $f\colon A \to A/Ax$. Applying the functor *, we get the morphism in $\mathrm{D^b}(A)$:

$$f^*\colon \mathrm{RHom}_A(A/Ax,A) \to A.$$

Remember that $\mathrm{RHom}_A(A/Ax,A) \simeq A/xA\,[-1]$. Hence $H^j(f^*) = 0$ for all $j \in \mathbb{Z}$, although $f^* \neq 0$ since $f^{**} = f$.

Let us give an example of an object of a derived category which is not isomorphic to the direct sum of its cohomology objects (hence, a situation in which Corollary 7.1.9 does not apply).

Example 7.4.12. Let $\mathbf{k}$ be a field and let $A = \mathbf{k}[x_1,x_2]$. Define the A-modules

$$M' = A/(Ax_1 + Ax_2), \quad M = A/(Ax_1^2 + Ax_1x_2), \quad M'' = A/Ax_1.$$

There is an exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{7.4.3}$$

and this exact sequence does not split since x_1 kills M' and M'' but not M.

Recall the functor * of (7.4.2). We have $M'^* \simeq H^2(M'^*)[-2]$ and $M'''^* \simeq H^1(M'^*)[-1]$. The functor * applied to the exact sequence (7.4.3) gives rise to the long exact sequence

$$0 \to H^1(M''^*) \to H^1(M^*) \to 0 \to 0 \to H^2(M^*) \to H^2(M'^*) \to 0.$$

Hence $H^1(M^*)[-1] \simeq H^1(M''^*)[-1] \simeq M''^*$ and $H^2(M^*)[-2] \simeq H^2(M'^*)[-2] \simeq M'^*$. Assume for a while $M^* \simeq \oplus_j H^j(M^*)[-j]$. This implies $M^* \simeq M'^* \oplus M''^*$ hence (by applying again the functor *), $M \simeq M' \oplus M''$, which is a contradiction.

7.5 The Brown representability theorem

We shall follow the exposition of [KS06, § 10.5].

Definition 7.5.1. Let $\mathscr{D}$ be a triangulated category admitting small direct sums. A *system of t-generators* $\mathscr{F}$ in $\mathscr{D}$ is a small family of objects of $\mathscr{D}$ satisfying conditions (i) and (ii) below.

(i) For any $X \in \mathscr{D}$ with $\mathrm{Hom}_{\mathscr{D}}(C, X) \simeq 0$ for all $C \in \mathscr{F}$, we have $X \simeq 0$.

(ii) For any *countable* set I and any family $\{u_i \colon X_i \to Y_i\}_{i\in I}$ of morphisms in $\mathscr{D}$, the map $\mathrm{Hom}_{\mathscr{D}}(C, \oplus_i X_i) \xrightarrow{\oplus_i u_i} \mathrm{Hom}_{\mathscr{D}}(C, \oplus_i Y_i)$ vanishes for every $C \in \mathscr{F}$ as soon as $\mathrm{Hom}_{\mathscr{D}}(C, X_i) \xrightarrow{u_i} \mathrm{Hom}_{\mathscr{D}}(C, Y_i)$ vanishes for every $i \in I$ and every $C \in \mathscr{F}$.

What we call below the Brown representability Theorem is in fact a corollary of such a theorem. See [KS06, Cor. 10.5.3].

Theorem 7.5.2. (The Brown representability Theorem.) *Let $\mathscr{D}$ be a triangulated category admitting small direct sums and a system of t-generators. Let $F \colon \mathscr{D} \to \mathscr{D}'$ be a triangulated functor of triangulated categories and assume that F commutes with small direct sums. Then F admits a right adjoint G and G is triangulated.*

Recall Definition 5.4.3 of a Grothendieck category and also recall that such a definition relies on the notion of universe. Hence, all categories in the sequel belong to a given universe $\mathscr{U}$.

We shall apply Theorem 7.5.2 in the particular case of derived categories. Recall Definition **??**.

Theorem 7.5.3. (see [KS06, Th. 14.2.1]) *Let $\mathscr{C}$ be a Grothendieck abelian category. Then $\mathrm{D}(\mathscr{C})$ admits small direct sums and a system of t-generators.*

Note that the existence of small direct sums follows from Proposition 6.4.9.

From now on, we shall follow [GS16, § 2.3].

Lemma 7.5.4. *Let $\mathscr{C}$ be a Grothendieck category and let $d \in \mathbb{Z}$. Then the cohomology functor H^d and the truncation functors $\tau^{\leq d}$ and $\tau^{\geq d}$ commute with small direct sums in $\mathrm{D}(\mathscr{C})$. In other words, if $\{X_i\}_{i\in I}$ is a small family of objects of $\mathrm{D}(\mathscr{C})$, then*

$$\bigoplus_i \tau^{\leq d} X_i \xrightarrow{\sim} \tau^{\leq d}(\bigoplus_i X_i) \tag{7.5.1}$$

and similarly with $\tau^{\geq d}$ and H^d.

Proof. (i) Let us treat first the functor H^d. Recall that $Q \colon \mathrm{K}(\mathscr{C}) \to \mathrm{D}(\mathscr{C})$ denotes the localization functor and Q commutes with small direct sums by Proposition 6.4.9. Let us denote for a while by $\widetilde{H}^d \colon \mathrm{K}(\mathscr{C}) \to \mathscr{C}$ the cohomology functor usually denoted by H^d. Then $\widetilde{H}^d \simeq H^d \circ Q$.

Let $\{X_i\}_i$ be a small family of objects in $\mathrm{K}(\mathscr{C})$. Then

$$\begin{aligned} H^d(\oplus_i Q(X_i)) &\simeq H^d(Q(\oplus_i X_i)) \simeq \widetilde{H}^d(\oplus_i X_i) \\ &\simeq \oplus_i \widetilde{H}^d(X_i) \simeq \oplus_i H^d(Q(X_i))). \end{aligned}$$

(ii) The morphism in (7.5.1) is well-defined and it is enough to check that it induces an isomorphism on the cohomology. This follows from (i) since for any object $Y \in \mathrm{D}(\mathscr{C})$, $H^j(\tau^{\leq d} Y)$ is either 0 or $H^j(Y)$. □

Lemma 7.5.5. *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Let I be a small category. Assume*

(i) *I is either filtrant or discrete,*

(ii) *ρ commutes with inductive limits indexed by I,*

(iii) *inductive limits indexed by I of injective objects in $\mathscr{C}$ are acyclic for the functor ρ.*

Then for all $j \in \mathbb{Z}$, the functor $R^j\rho\colon \mathscr{C} \to \mathscr{C}'$ commutes with inductive limits indexed by I.

Proof. Let $\alpha\colon I \to \mathscr{C}$ be a functor. Denote by $\mathscr{I}$ the full additive subcategory of $\mathscr{C}$ consisting of injective objects. It follows for example from [KS06, Cor. 9.6.6] that there exists a functor $\psi\colon I \to \mathscr{I}$ and a morphism of functors $\alpha \to \psi$ such that for each $i \in I$, $\alpha(i) \to \psi(i)$ is a monomorphism. Therefore one can construct a functor $\Psi\colon I \to \mathrm{C}^+(\mathscr{I})$ and a morphism of functor $\alpha \to \Psi$ such that for each $i \in I$, $\alpha(i) \to \Psi(i)$ is a quasi-isomorphism. Set $X_i = \alpha(i)$ and $G_i^\bullet = \Psi(i)$. We get a qis $X_i \to G_i^\bullet$, hence a qis

$$\operatorname*{colim}_i X_i \to \operatorname*{colim}_i G_i^\bullet .$$

On the other hand, we have

$$\begin{aligned}\operatorname*{colim}_i R^j\rho(X_i) &\simeq \operatorname*{colim}_i H^j(\rho(G_i^\bullet))\\ &\simeq H^j\rho(\operatorname*{colim}_i G_i^\bullet)\end{aligned}$$

where the second isomorphism follows from the fact that H^j commutes with direct sums and with filtrant inductive limits. Then the result follows from hypothesis (iii). □

Lemma 7.5.6. *We make the same hypothesis as in Lemma 7.5.5. Let $-\infty < a \leq b < \infty$, let I be a small set and let $X_i \in \mathrm{D}^{[a,b]}(\mathscr{C})$. Then*

$$\bigoplus_i R\rho(X_i) \xrightarrow{\sim} R\rho(\bigoplus_i X_i). \tag{7.5.2}$$

Proof. The morphism in (7.5.2) is well-defined and we have to prove it is an isomorphism. If $b = a$, the result follows from Lemma 7.5.5. The general case is deduced by induction on $b - a$ by considering the distinguished triangles

$$H^a(X_i)\,[-a] \to X_i \to \tau^{\geq a+1}X_i \xrightarrow{+1} .$$

□

Theorem 7.5.7. (see [GS16, Prop. 2.21]) *Let $\mathscr{C}$ and $\mathscr{C}'$ be two Grothendieck categories and let $\rho\colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor. Assume that*

(i) *ρ has finite cohomological dimension,*

(ii) *ρ commutes with small direct sums,*

(iii) *small direct sums of injective objects in* $\mathscr{C}$ *are acyclic for the functor* ρ.

Then

(a) *the functor* $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ *commutes with small direct sums,*

(b) *the functor* $R\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ *admits a right adjoint* $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$,

(c) *the functor* $\rho^!$ *induces a functor* $\rho^!\colon \mathrm{D}^+(\mathscr{C}') \to \mathrm{D}^+(\mathscr{C})$.

Proof. (a) Let $\{X_i\}_{i\in I}$ be a family of objects of $\mathrm{D}(\mathscr{C})$. It is enough to check that the natural morphism in $\mathrm{D}(\mathscr{C}')$

$$\bigoplus_{i\in I} R\rho(X_i) \to R\rho(\bigoplus_{i\in I} X_i) \tag{7.5.3}$$

induces an isomorphism on the cohomology groups. Assume that ρ has cohomological dimension $\leq d$. For $X \in \mathrm{D}(\mathscr{C})$ and for $j \in \mathbb{Z}$, we have

$$\tau^{\geq j} R\rho(X) \simeq \tau^{\geq j} R\rho(\tau^{\geq j-d-1} X).$$

The functor ρ being left exact we get for $k \geq j$:

$$H^k R\rho(X) \simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X). \tag{7.5.4}$$

We have the sequence of isomorphisms:

$$\begin{aligned} H^k R\rho(\bigoplus_i X_i) &\simeq H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} \bigoplus_i X_i) \simeq H^k R\rho(\bigoplus_i \tau^{\leq k}\tau^{\geq j-d-1} X_i) \\ &\simeq \bigoplus_i H^k R\rho(\tau^{\leq k}\tau^{\geq j-d-1} X_i) \simeq \bigoplus_i H^k R\rho(X_i). \end{aligned}$$

The first and last isomorphisms follow from (7.5.4).
The second isomorphism follows from Lemma 7.5.4.
The third isomorphism follows from Lemma 7.5.6.
(b) follows from (a) and the Brown representability theorem 7.5.2.

(c) This follows from hypothesis (i) and (the well-known) Lemma 7.5.8 below. □

Lemma 7.5.8. *Let* $\rho\colon \mathscr{C} \to \mathscr{C}'$ *be a left exact functor between two Grothendieck categories. Assume that* $\rho\colon \mathrm{D}(\mathscr{C}) \to \mathrm{D}(\mathscr{C}')$ *admits a right adjoint* $\rho^!\colon \mathrm{D}(\mathscr{C}') \to \mathrm{D}(\mathscr{C})$ *and assume moreover that* ρ *has finite cohomological dimension. Then the functor* $\rho^!$ *sends* $\mathrm{D}^+(\mathscr{C}')$ *to* $\mathrm{D}^+(\mathscr{C})$.

Proof. By the hypothesis, we have for $X \in \mathrm{D}(\mathscr{C})$ and $Y \in \mathrm{D}(\mathscr{C}')$

$$\mathrm{Hom}_{\mathrm{D}(\mathscr{C}')}(\rho(X), Y) \simeq \mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)).$$

Assume that the cohomological dimension of the functor ρ is $\leq r$. Let $Y \in \mathrm{D}^{\geq 0}(\mathscr{C}')$. Then (using Exercise 7.8) $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, \rho^!(Y)) \simeq 0$ for all $X \in \mathrm{D}^{<-r}(\mathscr{C})$. This implies that $\rho^!(Y) \in \mathrm{D}^{\geq -r}(\mathscr{C}')$. □

Exercises to Chapter 7

Exercise 7.1. Let $\mathscr{C}$ be an abelian category with enough injectives. Prove that the two conditions below are equivalent.

(i) For all X and Y in $\mathscr{C}$, $\mathrm{Ext}^j_{\mathscr{C}}(X,Y)\simeq 0$ for all $j>n$.

(ii) For all X in $\mathscr{C}$, there exists an exact sequence $0\to X\to X^0\to\cdots\to X^n\to 0$, with the X^j's injective.

In such a situation, one says that $\mathscr{C}$ has homological dimension $\leq n$ and one writes $\mathrm{dh}(\mathscr{C})\leq n$.

(iii) Assume moreover that $\mathscr{C}$ has enough projectives. Prove that (i) is equivalent to: for all X in $\mathscr{C}$, there exists an exact sequence $0\to X^n\to\cdots\to X^0\to X\to 0$, with the X^j's projective.

Exercise 7.2. Let $\mathscr{C}$ be an abelian category with enough injective and such that $\mathrm{dh}(\mathscr{C})\leq 1$. Let $F\colon\mathscr{C}\to\mathscr{C}'$ be a left exact functor and let $X\in\mathrm{D}^+(\mathscr{C})$.

(i) Construct an isomorphism $H^k(RF(X))\simeq F(H^k(X))\oplus R^1F(H^{k-1}(X))$.

(ii) Recall that $\mathrm{dh}(\mathrm{Mod}(\mathbb{Z}))=1$. Let $X\in\mathrm{D}^-(\mathbb{Z})$, and let $M\in\mathrm{Mod}(\mathbb{Z})$. Deduce the isomorphism

$$H^k(X\overset{\mathrm{L}}{\otimes}M)\simeq(H^k(X)\otimes M)\oplus\mathrm{Tor}^{-1}_{\mathbb{Z}}(H^{k+1}(X),M).$$

Exercise 7.3. Let $F\colon\mathscr{C}\to\mathscr{C}'$ be a left exact functor of abelian categories and let $\mathscr{J}$ be an F-injective additive subcategory of $\mathscr{C}$. Assume that F has cohomological dimension $\leq n$. Prove that for any $X\in\mathscr{C}$ there exists an exact sequence $0\to X\to X^0\to\cdots\to X^n\to 0$, with the X^j's in $\mathscr{J}$.

Exercise 7.4. Let $\mathscr{C}$ be an abelian category with enough injectives and let $0\to X'\to X\to X''\to 0$ be an exact sequence in $\mathscr{C}$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(X'',X')\simeq 0$, prove that the sequence splits.

Exercise 7.5. Let $\mathscr{C}$ be an abelian category and let $X\to Y\to Z\xrightarrow{+1}$ be a d.t. in $\mathrm{D}(\mathscr{C})$. Assuming that $\mathrm{Ext}^1_{\mathscr{C}}(Z,X)\simeq 0$, prove that $Y\simeq X\oplus Z$. (Hint: use Exercise 6.3.)

Exercise 7.6. Let $\mathscr{C}$ be an abelian category, let $X\in\mathrm{D}^{\mathrm{b}}(\mathscr{C})$ and let $a<b\in\mathbb{Z}$. Assume that $H^j(X)\simeq 0$ for $j\neq a,b$ and $\mathrm{Ext}^{b-a+1}_{\mathscr{C}}(H^b(X),H^a(X))\simeq 0$. Prove the isomorphism $X\simeq H^a(X)[-a]\oplus H^b(X)[-b]$. (Hint: use Exercise 7.5 and the d.t. in 7.1.3.

Exercise 7.7. We follow the notations of Exercise 5.9. Hence, $\mathbf{k}$ is a field of characteristic 0 and $W:=W_n(\mathbf{k})$ is the Weyl algebra in n variables. Let $1\leq p\leq n$ and consider the left ideal

$$I_p=W\cdot x_1+\cdots+W\cdot x_p+W\cdot\partial_{p+1}+\cdots+W\cdot\partial_n.$$

Define similarly the right ideal

$$J_p=x_1\cdot W+\cdots+x_p\cdot W+\partial_{p+1}\cdot W+\cdots+\partial_n\cdot W.$$

For $1\leq p\leq q\leq n$, calculate $\mathrm{RHom}_W(W/I_p,W/I_q)$ and $W/J_q\overset{\mathrm{L}}{\otimes}_W W/I_p$.

Exercise 7.8. Let $\mathscr{C}$ be an abelian category.

(a) Let $X \in \mathrm{D}^{<0}(\mathscr{C})$ and $Y \in \mathrm{D}^{\geq 0}(\mathscr{C})$. Prove that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y) \simeq 0$.

(b) Conversely, let $Y \in \mathrm{D}(\mathscr{C})$ and assume that $\mathrm{Hom}_{\mathrm{D}(\mathscr{C})}(X, Y) \simeq 0$ for all $X \in \mathrm{D}^{<0}(\mathscr{C})$. Prove that $Y \in \mathrm{D}^{\geq 0}(\mathscr{C})$.

Exercise 7.9. Let $F \colon \mathscr{C} \to \mathscr{C}'$ be a left exact functor of abelian categories and assume that F has a right derived functor and has cohomological dimension $\leq d$. Denote by $\mathscr{J}$ the additive subcategory of $\mathscr{C}$ consisting of F-injective objects and consider an exact sequence $0 \to X \to X^0 \to X^1 \to \cdots \to X^d \to 0$ with $X^j \in \mathscr{J}$ for $0 \leq j < d$. Prove that $X^d \in \mathscr{J}$.
(Hint: decompose the exact sequence into short exact sequences $0 \to Z^{j-1} \to X^j \to Z^j \to 0$ with $Z^{-1} = X$ and show that $R^jF(Z^k) \simeq 0$ for $j > d - k$.)

Chapter 8

Sheaves on sites

A presheaf on a topological space X with values in a category $\mathscr{A}$ is nothing but a contravariant functor defined on the category of the open subsets of X with values in $\mathscr{A}$. It is thus natural to extend this definition, as did Grothendieck. A presheaf on a small category $\mathscr{C}_X$ is nothing but a functor defined on $\mathscr{C}_X^{\mathrm{op}}$ and $\mathscr{C}_X$ is called a presite.

A site X is a presite endowed with a "Grothendieck topology". For each $U \in \mathscr{C}_X$, one is given a family of "coverings" of U, a covering being a family of morphisms $V \to U$, this family playing the role of usual coverings on topological spaces. The theory is much easier when assuming, as we do here, that the category $\mathscr{C}_X$ admits finite products and fiber products.

This chapter starts with a rapid overview of classical abelian sheaves on topological spaces. Then we study presheaves on presites, before introducing Grothendieck topologies and sheaves on sites. Under suitable hypotheses on the category $\mathscr{A}$, we construct the sheaf associated with a presheaf and study the category $\mathrm{Sh}(X, \mathscr{A})$ of sheaves on X with values in $\mathscr{A}$. We also study the operations of direct and inverse images as well as those of restriction and extension of sheaves. In the course of the study, we introduce locally constant sheaves. They are of fundamental importance in mathematics and are a first step towards constructible sheaves which will be treated later.

Finally, we glue sheaves, that is, given a covering of X and sheaves defined on the open sets of this coverings satisfying a natural cocycle condition, we prove the existence and unicity of a sheaf on X locally isomorphic to these locally defined sheaves. In other words, we show that the prestack of sheaves on X is a stack.

Caution. As already mentioned, we fix a universe $\mathscr{U}$ and, unless otherwise specified, a category means a $\mathscr{U}$-category. Moreover, all rings, topological spaces, etc., are supposed to be small.

Some references. As already mentioned in the introduction, classical sheaf theory was first exposed in the book of Roger Godement [God58] then in [Bre67]. For an approach in the language of derived categories, see [Ive86, GM96, KS90]. Sheaves on Grothendieck topologies are exposed in [SGA4] and [KS06]. A short presentation in case of the étale topology is given in [Tam94].

8.1 Abelian sheaves on topological spaces: a short introduction

In this introductory section, we shall define sheaves of $\mathbf{k}$-modules on a topological space.

Let X be a topological space. The family of open subsets of X is ordered by inclusion and we denote by Op_X the associated category. Hence:

$$\mathrm{Hom}_{\mathrm{Op}_X}(U,V)=\begin{cases}\{\mathrm{pt}\} & \text{if } U\subset V,\\ \varnothing & \text{otherwise.}\end{cases}$$

Note that the category Op_X admits a terminal object, namely X, and finite products, namely $U\times V=U\cap V$. Indeed, any pair of morphisms $(W\to U, W\to V)$, (that is, inclusions $(W\subset U, W\subset V)$) factors uniquely through a morphism $W\to U\cap V$.

Definition 8.1.1. (a) A presheaf F on X of $\mathbf{k}$-modules is a functor $\mathrm{Op}_X^{\mathrm{op}}\to\mathrm{Mod}(\mathbf{k})$.

(b) A morphism of presheaves is a morphism of such functors.

(c) One denotes by $\mathrm{PSh}(\mathbf{k}_X)$ the category of presheaves of $\mathbf{k}$-modules.

In other words, a presheaf F associates to each open set $U\subset X$ a $\mathbf{k}$-module $F(U)$ and to each open inclusion $U\subset V$ a linear map $F(V)\to F(U)$, often called the restriction map, these maps being compatible, that is, for $U\subset V\subset W$, the composition of the restriction maps $F(W)\to F(V)\to F(U)$ is the restriction map $F(W)\to F(U)$ and the restriction $F(U)\to F(U)$ is the identity map.

A morphism of presheaves $\varphi\colon F\to G$ is the data for any open set U of a map $\varphi(U)\colon F(U)\to G(U)$ such that for any open inclusion $U\subset V$, the diagram below commutes:

(8.1.1)
$$\begin{tikzcd} F(V) \arrow[r, "\varphi(V)"] \arrow[d] & G(V) \arrow[d] \\ F(U) \arrow[r, "\varphi(U)"] & G(U). \end{tikzcd}$$

- If $s\in F(V)$, one says that s is a section of F on V and if U is an open subset of V, one denotes by $s|_U$ its image by the restriction map.
- If $U\in\mathrm{Op}_X$ and $F\in\mathrm{PSh}(\mathbf{k}_X)$, one defines the presheaf $F|_U$ on U, the restriction of F to U, by setting for $W\in\mathrm{Op}_U$, $(F|_U)(W)=F(W)$.

Examples 8.1.2. (i) Let $M\in\mathrm{Mod}(\mathbf{k})$. The correspondence $U\mapsto M$ is a presheaf, called the constant presheaf on X with fiber M.
(ii) Let $\mathscr{C}^0(U)$ denote the $\mathbb{C}$-vector space of $\mathbb{C}$-valued continuous functions on U. Then $U\mapsto\mathscr{C}^0(U)$ (with the usual restriction morphisms) is a presheaf of $\mathbb{C}$-vector spaces, denoted $\mathscr{C}^0_X$. If $\varphi\in\mathscr{C}^0(X)$, multiplication by φ is an endomorphism of the presheaf $\mathscr{C}^0_X$.

Consider a family $\mathscr{U}:=\{U_i\}_{i\in I}$ of open subsets of X indexed by a set I. One says that $\mathscr{U}$ is an open covering of U if $U_i\subset U$ for all i and $\bigcup_i U_i=U$.

Let F be a presheaf on X and consider the two conditions below.

S1 For any open subset $U \subset X$, any open covering $\{U_i\}_{i\in I}$ of U, any $s \in F(U)$ satisfying $s|_{U_i} = 0$ for all i, one has $s = 0$.

S2 For any open subset $U \subset X$, any open covering $\{U_i\}_{i\in I}$ of U, any family $\{s_i \in F(U_i)\}_{i\in I}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there exists $s \in F(U)$ with $s|_{U_i} = s_i$ for all i.

Definition 8.1.3. (i) One says that the presheaf F is separated if it satisfies S1. One says that F is a sheaf if it satisfies S1 and S2.

(ii) One denotes by $\mathrm{Mod}(\mathbf{k}_X)$ the full subcategory of $\mathrm{PSh}(\mathbf{k}_X)$ whose objects are sheaves and by $\iota_X\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{PSh}(\mathbf{k}_X)$ the forgetful functor.

(iii) One writes $\mathrm{Hom}_{\mathbf{k}_X}(\,\bullet\,,\,\bullet\,)$ instead of $\mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}(\,\bullet\,,\,\bullet\,)$.

Notation 8.1.4. Let F be a sheaf of $\mathbf{k}$-modules on X.

(i) One defines its support, denoted by $\operatorname{supp} F$, as the complementary of the union of all open subsets U of X such that $F|_U = 0$.

(ii) Let $s \in F(U)$. One defines its support, denoted by $\operatorname{supp} s$, as the complementary in U of the union of all open subsets V of U such that $s|_V = 0$.

Example 8.1.5. Let X be a topological space. The presheaf $\mathscr{C}^0_X$ of $\mathbb{C}$-valued continuous functions is a sheaf.

Now, let $M \in \mathrm{Mod}(\mathbf{k})$. The presheaf of locally constant functions on X with values in M is a sheaf, called the constant sheaf with stalk M and denoted M_X. Note that if $X \neq \varnothing$, the constant presheaf with stalk M is not a sheaf except if $M = 0$.

Examples 8.1.6. On a manifold X we have the following classical sheaves.

(a) Let X be a real manifold of class C^∞.

$\mathscr{C}^\infty_X$: the sheaf of complex valued functions of class C^∞,
$\mathscr{D}b_X$: the sheaf of complex valued Schwartz's distributions,
$\mathscr{C}^{\infty,(p)}_X$: the sheaf of p-forms of class $\mathscr{C}^\infty$, $\mathscr{D}b^{(p)}_X$: the sheaf of p-forms with distributions as coefficients.

(b) Let X be a real manifold of class C^ω, that is, a real analytic manifold.

$\mathscr{A}_X$, or $\mathscr{C}^\omega_X$: the sheaf of complex valued real analytic functions,
$\mathscr{A}^{(p)}_X$ or $\mathscr{C}^{\omega,(p)}_X$: the sheaf of p-forms of class C^ω,
$\mathscr{B}_M$: the sheaf of Sato's hyperfunctions,
$\mathscr{B}^{(p)}_M$: the sheaf of p-forms with hyperfunctions as coefficients.

(c) Let X is a complex manifold.

$\mathscr{O}_X$: the sheaf holomorphic functions,
Ω^p_X: the sheaf of holomorphic p-forms,
$\mathscr{D}_X$: the sheaf of (finite order) differential operators with coefficients in $\mathscr{O}_X$.

Examples 8.1.7. (a) On a topological space X, the presheaf $U \mapsto \mathscr{C}^{0,b}_X(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property and axiom S2 is not satisfied.

(b) Let $X = \mathbb{C}$ and denote by z the holomorphic coordinate. The holomorphic derivation $\frac{\partial}{\partial z}$ is a endomorphism of the sheaf $\mathscr{O}_X$. Consider the presheaf $F\colon U \mapsto \mathscr{O}(U)/\frac{\partial}{\partial z}\mathscr{O}(U)$, that is, the presheaf $\operatorname{Coker}(\frac{\partial}{\partial z}\colon \mathscr{O}_X \to \mathscr{O}_X)$. For U an open disk, $F(U) \simeq 0$ since the equation $\frac{\partial}{\partial z}f = g$ is always solvable. However, if $U = \mathbb{C}\setminus\{0\}$, $F(U) \neq 0$. Hence the presheaf F does not satisfy axiom S1.

(c) If F is a sheaf on X and U is open, then $F|_U$ is sheaf on U.

Definition 8.1.8. Let $x \in X$, and let I_x denote the full subcategory of Op_X consisting of open neighborhoods of x. For a presheaf F on X, one sets:

$$F_x = \operatorname*{colim}_{U \in I_x^{\mathrm{op}}} F(U). \tag{8.1.2}$$

One calls F_x the stalk of F at x.

Proposition 8.1.9. *The functor $F \mapsto F_x$ from $\mathrm{PSh}(\mathbf{k}_X)$ to $\mathrm{Mod}(\mathbf{k})$ is exact.*

Proof. The functor $F \mapsto F_x$ is the composition

$$\mathrm{PSh}(\mathbf{k}_X) = \mathrm{Fct}(\mathrm{Op}_X^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Fct}(I_x^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \xrightarrow{\mathrm{colim}} \mathrm{Mod}(\mathbf{k}).$$

The first functor associates to a presheaf F its restriction to the category I_x^{op}. It is clearly exact. Since $U, V \in I_x$ implies $U \cap V \in I_x$, the category I_x^{op} is filtered and it follows from Corollary 2.6.7 that, in this situation, the functor colim is exact. □

Let $x \in U$ and let $s \in F(U)$. The image $s_x \in F_x$ of s is called the germ of s at x.

Since I_x^{op} is filtered, a germ $s_x \in F_x$ is represented by a section $s \in F(U)$ for some open neighborhood U of x, and for $s \in F(U), t \in F(V)$, $s_x = t_x$ means that there exists an open neighborhood W of x with $W \subset U \cap V$ such that $\rho_{WU}(s) = \rho_{WV}(t)$. (See also Example 2.6.12.)

8.2 Presites and presheaves

Many proofs in this section are particularly tedious and may be skipped, at least in a first reading.

Definition 8.2.1. (i) A presite X is a small category $\mathscr{C}_X$.

(ii) A morphism of presites $f\colon X \to Y$ is a functor $f^t\colon \mathscr{C}_Y \to \mathscr{C}_X$.

In the sequel, we shall say that a presite X has a property $\mathscr{P}$ if the category $\mathscr{C}_X$ has the property $\mathscr{P}$.

For example, we say that X has a terminal object if so has $\mathscr{C}_X$. In such a case, we denote this object by X.

We denote by X^{op} the presite associated with the category $\mathscr{C}_X^{\mathrm{op}}$.

We denote by $\widehat{X}$ the presite associated with the category $\mathscr{C}_X^\wedge$.

Example 8.2.2. We shall identify a topological space X to the presite associated with the category Op_X.

Let $f\colon X \to Y$ be a continuous map of topological spaces. It defines a morphism of presites by setting

$$f^t(V) := f^{-1}(V) \text{ for } V \in \mathrm{Op}_Y.$$

In particular, for U open in X, there are natural morphisms of presites

$$\begin{aligned} &i_U\colon U \to X, \mathrm{Op}_X \ni V \mapsto (U \cap V) \in \mathrm{Op}_U,\\ &j_U\colon X \to U, \mathrm{Op}_U \ni V \mapsto V \in \mathrm{Op}_X. \end{aligned} \tag{8.2.1}$$

This example already shows that, when considering topological spaces as presites, we get morphisms of presites which do not correspond to any morphism of topological spaces.

Definition 8.2.3. Let $\mathscr{A}$ be a category.

(i) An $\mathscr{A}$-valued presheaf F on a presite X is a functor $F\colon \mathscr{C}_X^{\mathrm{op}} \to \mathscr{A}$.

(ii) One denotes by $\mathrm{PSh}(X, \mathscr{A})$ the (big) category of presheaves on X with values in $\mathscr{A}$. In other words, $\mathrm{PSh}(X, \mathscr{A}) = \mathrm{Fct}(\mathscr{C}_X^{\mathrm{op}}, \mathscr{A})$.

(iii) One sets $\mathrm{PSh}(X) := \mathrm{PSh}(X, \mathbf{Set})$. In other words, $\mathrm{PSh}(X) = \mathscr{C}_X^\wedge$.

(iv) One sets $\mathrm{PSh}(\mathbf{k}_X) := \mathrm{PSh}(X, \mathrm{Mod}(\mathbf{k}))$ and called an object of $\mathrm{PSh}(\mathbf{k}_X)$ a $\mathbf{k}$-abelian presheaf, or an abelian presheaf, for short.

- A presheaf F on X associates to each object $U \in \mathscr{C}_X$ an object $F(U)$ of $\mathscr{A}$, and to each morphism $u\colon U \to V$, a morphism $\rho_u\colon F(V) \to F(U)$, such that for $v\colon V \to W$, one has:

$$\rho_{\mathrm{id}_U} = \mathrm{id}_{F(U)}, \qquad \rho_{v \circ u} = \rho_u \circ \rho_v.$$

- The morphism ρ_u is called a restriction morphism. When there is no risk of confusion, we shall not write it.
- A morphism of presheaves $\varphi\colon F \to G$ is thus the data for any $U \in \mathscr{C}_X$ of a morphism $\varphi(U)\colon F(U) \to G(U)$ such that for any morphism $V \to U$, Diagram 8.1.1 commutes.
- The category $\mathrm{PSh}(X, \mathscr{A})$ inherits of most all properties of the category $\mathscr{A}$. For example, if $\mathscr{A}$ admits small limits (resp. colimits) then so does $\mathrm{PSh}(X, \mathscr{A})$. If $\mathscr{A}$ is abelian, then $\mathrm{PSh}(X, \mathscr{A})$ is abelian.
- If $\mathscr{A}$ is a subcategory of $\mathbf{Set}$ (or more generally of $\mathbf{Set}^I$ for a small set I) and $U \in \mathscr{C}_X$, an element s of $F(U)$ is called a section of F on U.
- In view of the Yoneda lemma, the functor

$$\mathrm{h}_X\colon \mathscr{C}_X \hookrightarrow \mathrm{PSh}(X), \quad U \mapsto \mathrm{Hom}_{\mathscr{C}_X}(\,\bullet\,, U)$$

is fully faithful. One shall be aware that, when $\mathscr{C}_X$ admits limits or colimits, the functor h_X commutes with limits but not with colimits in general. (See Exercise 2.10.)

The functor $\Gamma(U;\bullet)$

For $U \in \mathscr{C}_X$, one defines the functor $\Gamma(U;\bullet)\colon \mathrm{PSh}(X,\mathscr{A}) \to \mathscr{A}$ by setting for $F \in \mathrm{PSh}(X,\mathscr{A})$:

$$\Gamma(U;F) = F(U). \tag{8.2.2}$$

Assuming that $\mathscr{A}$ admits limits, one also sets

$$\Gamma(X;F) = \lim_{U} F(U). \tag{8.2.3}$$

Of course, if $\mathscr{C}_X$ admits a terminal object X, equations (8.2.3) and (8.2.2) are compatible.

The functor $\Gamma(U;\bullet)$ commutes with limits and colimits as soon as $\mathscr{A}$ admits such limits. For example, if $\mathscr{A}$ is an abelian category and $\varphi\colon F \to G$ is a morphism of presheaves, then $(\ker\varphi)(U) \simeq \ker\varphi(U)$ and $(\mathrm{Coker}\,\varphi)(U) \simeq \mathrm{Coker}\,\varphi(U)$, where $\varphi(U)\colon F(U) \to G(U)$.

Examples 8.2.4. (i) Let $M \in \mathscr{A}$. The correspondence $U \mapsto M$ is a presheaf, called the constant presheaf on X with fiber M.
(ii) When X is a topological space and $\mathscr{A} = \mathrm{Mod}(\mathbb{C})$, we have already encountered the presheaf $\mathscr{C}^0(U)$ of $\mathbb{C}$-valued continuous functions on X.

8.3 Operations on presheaves

In this section, we shall consider a category $\mathscr{A}$ satisfying

$$\mathscr{A} \text{ admits small limits and colimits.} \tag{8.3.1}$$

Direct and inverse images of presheaves

Recall that a morphism of presites $f\colon X \to Y$ is a functor $f^t\colon \mathscr{C}_Y \to \mathscr{C}_X$. We shall use Definition 1.4.5.

Definition 8.3.1. Consider a morphism of presites $f\colon X \to Y$.

(i) Let $F \in \mathrm{PSh}(X,\mathscr{A})$. One defines $f_*F \in \mathrm{PSh}(Y,\mathscr{A})$, the direct image of F by f, by setting for $V \in \mathscr{C}_Y$: $f_*F(V) = F(f^t(V))$.

(ii) Let $G \in \mathrm{PSh}(Y,\mathscr{A})$. One defines $f^\dagger G \in \mathrm{PSh}(X,\mathscr{A})$ and $f^\ddagger G \in \mathrm{PSh}(X,\mathscr{A})$ by setting for $U \in \mathscr{C}_X$:

$$\begin{aligned} f^\dagger G(U) &= \operatorname*{colim}_{(U\to f^t(V))\in(\mathscr{C}_Y^U)^{\mathrm{op}}} G(V),\\ f^\ddagger G(U) &= \lim_{(f^t(V)\to U)\in(\mathscr{C}_Y)_U} G(V). \end{aligned}$$

Note that $f^\dagger G$ is a well defined presheaf on X. Indeed, consider a morphism $u\colon U \to U'$ in $\mathscr{C}_X$. The morphism $f^\dagger G(U') \to f^\dagger G(U)$ is given by:

$$f^\dagger G(U') = \operatorname*{colim}_{(U'\to f^t(V'))} G(V') \to \operatorname*{colim}_{(U\to f^t(V))} G(V).$$

There is a similar remark with $f^\ddagger G$.

Example 8.3.2. Assume that X and Y are topological spaces as in Example 8.2.2. Then for $U \in \mathrm{Op}_X$,

$$\begin{aligned} f^\dagger G(U) &= \operatorname*{colim}_{U \subset f^{-1}V} G(V), \\ f^\ddagger G(U) &= \lim_{f^{-1}V \subset U} G(V). \end{aligned}$$

Theorem 8.3.3. *Let $f\colon X \to Y$ be a morphism of presites.*

(i) *We have a pair of adjoint functors $(f^\dagger, f_*)$:*

$$f^\dagger\colon \mathrm{PSh}(Y, \mathscr{A}) \rightleftarrows \mathrm{PSh}(X, \mathscr{A})\colon f_*.$$

In other words, we have an isomorphism of bifunctors

$$\operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(f^\dagger(\,\bullet\,), \,\bullet\,) \simeq \operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(\,\bullet\,, f_*(\,\bullet\,)). \tag{8.3.2}$$

(ii) *Similarly, we have a pair of adjoint functors $(f_*, f^\ddagger)$:*

$$f_*\colon \mathrm{PSh}(X, \mathscr{A}) \rightleftarrows \mathrm{PSh}(Y, \mathscr{A})\colon f^\ddagger.$$

In other words, we have an isomorphism of bifunctors

$$\operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(f_*(\,\bullet\,), \,\bullet\,) \simeq \operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(\,\bullet\,, f^\ddagger(\,\bullet\,)). \tag{8.3.3}$$

Proof. Since (i) and (ii) are equivalent by reversing the arrows, that is, by considering the morphism of presites $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$, we shall only prove (i).

(a) First, we construct a map

$$\Phi\colon \operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(G, f_*F) \to \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(f^\dagger G, F).$$

Let $\theta \in \operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(G, f_*F)$ and let $U \in \mathscr{C}_X$. For $V \in \mathscr{C}_Y$ and a morphism $U \to f^t(V)$, the composition

$$G(V) \xrightarrow{\theta(V)} F(f^t(V)) \to F(U)$$

gives a morphism $\Phi(\theta)(U)\colon \operatorname*{colim}_{U \to f^t(V)} G(V) \to F(U)$. The morphism $\Phi(\theta)(U)$ is functorial in U, that is, for any morphism $U' \to U$ in $\mathscr{C}_X$, the diagram below commutes:

$$\begin{tikzcd} \operatorname*{colim}_{U \to f^t(V)} G(V) \arrow[r, "\Phi(\theta)(U)"] \arrow[d] & F(U) \arrow[d] \\ \operatorname*{colim}_{U' \to f^t(V')} G(V') \arrow[r, "\Phi(\theta)(U')"] & F(U'). \end{tikzcd}$$

Therefore, the family of morphisms $\{\Phi(\theta)(U)\}_U$ defines the morphism $\Phi(\theta)$.

(b) Next, we construct a map

$$\Psi\colon \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(f^\dagger G, F) \to \operatorname{Hom}_{\mathrm{PSh}(Y,\mathscr{A})}(G, f_*F).$$

Let $\lambda \in \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(f^\dagger G, F)$ and let $V \in \mathscr{C}_Y$. The morphism

$$\lambda(f^tV)\colon f^\dagger G(f^tV) = \operatorname*{colim}_{f^tV \to W} G(W) \to F(f^tV)$$

together with the morphism $G(V) \to \operatorname*{colim}_{f^tV \to W} G(W)$ defines the morphism

$$\Psi(\lambda)(V)\colon G(V) \to F(f^tV) = f_*F(V).$$

The morphism $\Psi(\lambda)(V)$ is functorial in V and defines $\Psi(\lambda)$.
(c) The reader will check that Ψ and Φ are inverse one to each other. ☐

Proposition 8.3.4. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of presites. Then*

$$\begin{aligned}(g \circ f)_* &\simeq g_* \circ f_*,\\ (g \circ f)^\dagger &\simeq f^\dagger \circ g^\dagger,\\ (g \circ f)^\ddagger &\simeq f^\ddagger \circ g^\ddagger.\end{aligned}$$

Proof. The first isomorphism is obvious and the others follow by adjunction. ☐

Remark 8.3.5. The constructions of the functors $f^\dagger$ and $f^\ddagger$ are obtained by the so-called Kan extension of functors.

Restriction and extension of presheaves

Let X be a presite. We shall always make the hypothesis:

(8.3.4) the presite X admits products of two objects and fiber products.

Note that a category admits a terminal object and fiber products if and only if it admits finite projective limits. If a category $\mathscr{C}_X$ admits a terminal object X, then $U \times_X V \xrightarrow{\sim} U \times V$ for any $U, V \in \mathscr{C}_X$.

Notation 8.3.6. We do not ask the presite X satisfying (8.3.4) to have a terminal object. However, for $U_1, U_2 \in \mathscr{C}_X$, we shall denote by $U_1 \times_X U_2$ their product in $\mathscr{C}_X$.

For $U \in \mathscr{C}_X$, we set $\mathscr{C}_U := (\mathscr{C}_X)_U$ and we still denote by U the presite associated with the category $\mathscr{C}_U$. We shall use the two morphisms of presites $\mathrm{j}_U\colon X \to U$ and $i_U\colon U \to X$ associated with the functors j_U^t and i_U^t:

$$\begin{aligned}&\mathrm{j}_U\colon X \to U, \quad \mathrm{j}_U^t\colon \mathscr{C}_U \to \mathscr{C}_X, \quad (V \to U) \mapsto V,\\ &i_U\colon U \to X, \quad i_U^t\colon \mathscr{C}_X \to \mathscr{C}_U, \quad V \mapsto (U \times_X V \to U).\end{aligned} \tag{8.3.5}$$

Recall (8.2.1) for a description of the functors i_U and j_U in case of topological spaces.

Let $F \in \mathrm{PSh}(X, \mathscr{A})$. One sets

$$F|_U = \mathrm{j}_{U*}F$$

and one calls $F|_U$ the restriction of F to U.

Proposition 8.3.7. *Let $U \in \mathscr{C}_X$. One has the isomorphisms of functors of presheaves*

$$\mathrm{j}_{U*} \simeq i_U^\dagger, \quad \mathrm{j}_U^\ddagger \simeq i_{U*}.$$

Proof. Let $F \in \mathrm{PSh}(\mathbf{k}_X, \mathscr{A})$ and let $(V \to U) \in \mathscr{C}_U$. One has

$$\begin{aligned}(i_U^{\dagger} F)(V \to U) &\simeq \operatorname*{colim}_{W \to V \to U} F(W)\\ &\simeq F(V) \simeq (\mathrm{j}_{U*}F)(V \to U).\end{aligned}$$

The isomorphism $\mathrm{j}_U^{\ddagger} \simeq i_{U*}$ follows by adjunction. □

More generally, consider a morphism $s\colon V \to U$ in $\mathscr{C}_X$. It defines a functor

(8.3.6) $\mathrm{j}^t_{V \xrightarrow{s} U}\colon \mathscr{C}_V \to \mathscr{C}_U$, hence, a morphism of sites $\mathrm{j}_{V \xrightarrow{s} U}\colon U \to V$.

Proposition 8.3.8. *Let $U \in \mathscr{C}_X$ and $(V \to U) \in \mathscr{C}_U$. For $F \in \mathrm{PSh}(X, \mathscr{A})$ and $G \in \mathrm{PSh}(U, \mathscr{A})$, we have:*

(i) $(\mathrm{j}_{U*}F)(V \to U) \simeq F(V)$,

(ii) $\mathrm{j}_U^{\dagger}\, G(V) \simeq \coprod_{s \in \mathrm{Hom}_{\mathscr{C}_X}(V,U)} G(V \xrightarrow{s} U)$.

(iii) $\mathrm{j}_U^{\ddagger}\, G(V) \simeq G(U \times_X V \to U)$.

Proof. (i) is obvious.
(ii) By its definition,

$$\begin{aligned}\mathrm{j}_U^{\dagger}\, G(V) &\simeq \operatorname*{colim}_{(V \to \mathrm{j}_U^t(W \to U)) \in ((\mathscr{C}_U)^V)^{\mathrm{op}}} G(W \to U)\\ &\simeq \operatorname*{colim}_{V \to W \to U} G(W \to U)\\ &\simeq \operatorname*{colim}_{(s\colon V \to U) \in \mathrm{Hom}\,(V,U)} G(V \xrightarrow{s} U).\end{aligned}$$

Here, we use the fact that the category $\mathrm{Hom}_{\mathscr{C}_X}(V, U)$ is cofinal in $((\mathscr{C}_U)^V)^{\mathrm{op}}$. The result follows since the category $\mathrm{Hom}_{\mathscr{C}_X}(V, U)$ is discrete.
(iii) By its definition,

$$\begin{aligned}\mathrm{j}_U^{\ddagger}\, G(V) &\simeq \lim_{(\mathrm{j}_U^t(W \to U) \to V) \in (\mathscr{C}_U)_V} G(W \to U)\\ &\simeq \lim_{U \leftarrow W \to V} G(W \to U)\\ &\simeq G(U \times_X V \to U).\end{aligned}$$

Here, the last isomorphism follows from the fact that $U \times_X V \to U$ is a terminal object in $(\mathscr{C}_U)_V$. □

Internal hom

Recall Definition 1.4.6 for the definition of Mor_0.

Proposition 8.3.9. *Let $F, G \in \mathrm{PSh}(X, \mathscr{A})$. There is a natural isomorphism*

(8.3.7) $$\lambda\colon \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(F, G) \xrightarrow{\sim} \lim_{(U \to V) \in \mathrm{Mor}_0(\mathscr{C}_X)^{\mathrm{op}}} \mathrm{Hom}_{\mathscr{A}}(F(V), G(U)).$$

Proof. (i) First, we construct the map λ. Let $\varphi\colon F \to G$ be a morphism in $\mathrm{PSh}(X, \mathscr{A})$ and let $U \to V$ be a morphism in $\mathscr{C}_X$. The morphism $\varphi(U)\colon F(U) \to G(U)$ and the restriction morphism $F(V) \to F(U)$ define $\varphi_{U\to V}\colon F(V) \to G(U)$. Moreover a morphism $a\colon (U \to V) \to (U' \to V')$ in $\mathrm{Mor}_0(\mathscr{C}_X)$ defines a morphism

$$\varphi_a\colon \mathrm{Hom}_{\mathscr{A}}(F(V'), G(U')) \to \mathrm{Hom}_{\mathscr{A}}(F(V), G(U))$$

as follows. To $\varphi_{U'\to V'}\colon F(V') \to G(U')$, one associates the composition

$$\varphi_{U\to V}\colon F(V) \to F(V') \xrightarrow{\varphi_{U'\to V'}} G(U') \to G(U).$$

(ii) The map λ is injective. Indeed, $\lambda(\varphi) = \lambda(\psi)$ implies that $\varphi(U) = \psi(U)$ for all $U \in \mathscr{C}_X$.

(iii) The map λ is surjective. Let $\{\varphi(U \to V)\}_{U\to V} \in \varprojlim\limits_{U\to V} \mathrm{Hom}_{\mathscr{A}}(F(V), G(U))$. To a morphism $s\colon U \to V$ in $\mathscr{C}_X$, one associates the two morphisms in $\mathrm{Mor}_0(\mathscr{C}_X)$:

$$\begin{tikzcd}
U \arrow[r, "s"] \arrow[d] & V & U \arrow[r, "s"] \arrow[d, "s"'] & V \\
U \arrow[r] & U, \arrow[u, "s"'] & V \arrow[r] & V \arrow[u]
\end{tikzcd}$$

In the the diagram below, the two triangles commute. Hence, the square commutes.

$$\begin{tikzcd}
F(V) \arrow[r, "\varphi(V)"] \arrow[d] \arrow[dr, "\varphi(U\to V)" description] & G(V) \arrow[d] \\
F(U) \arrow[r, "\varphi(U)"'] & G(U).
\end{tikzcd} \tag{8.3.8}$$

Therefore, the family $\{\varphi(U \to V)\}_{U\to V}$ defines a morphism of functors $\varphi\colon F \to G$. □

Let $s\colon V \to U$ be a morphism in $\mathscr{C}_X$ and let $F, G \in \mathrm{PSh}(X, \mathscr{A})$. The restriction functor $\mathrm{j}_{V\xrightarrow{s}U*}\colon \mathrm{PSh}(U, \mathscr{A}) \to \mathrm{PSh}(V, \mathscr{A})$ defines the map

$$\mathrm{Hom}_{\mathrm{PSh}(U,\mathscr{A})}(F|_U, G|_U) \to \mathrm{Hom}_{\mathrm{PSh}(V,\mathscr{A})}(F|_V, G|_V).$$

Definition 8.3.10. Let $F, G \in \mathrm{PSh}(X, \mathscr{A})$. One denotes by $\mathscr{H}om(F, G)$ the presheaf of sets on X, $U \mapsto \mathrm{Hom}_{\mathrm{PSh}(U,\mathscr{A})}(F|_U, G|_U)$.

By its definition, we have for $U \in \mathscr{C}_X$:

$$\mathrm{Hom}_{\mathrm{PSh}(U,\mathscr{A})}(F|_U, G|_U) \simeq \mathscr{H}om(F, G)(U). \tag{8.3.9}$$

Note that in case $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, then $\mathscr{H}om(F, G)$ belongs to $\mathrm{Mod}(\mathbf{k}_X)$.

Tensor products

In this subsection, we assume that $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$.

Definition 8.3.11. Let $F_1, F_2 \in \mathrm{PSh}(\mathbf{k}_X)$. Their tensor product, denoted $F_1 \overset{\mathrm{psh}}{\otimes} F_2$ is the presheaf $U \mapsto F_1(U) \otimes F_2(U)$.

Proposition 8.3.12. *Let $F_i \in \mathrm{PSh}(\mathbf{k}_X)$, $(i = 1, 2, 3)$. There are natural isomorphisms:*

$$\begin{aligned}&\mathscr{H}om\,(F_1 \overset{\mathrm{psh}}{\otimes} F_2, F_3) \simeq \mathscr{H}om\,(F_1, \mathscr{H}om\,(F_2, F_3)),\\ &\mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F_1 \overset{\mathrm{psh}}{\otimes} F_2, F_3) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(\mathbf{k}_X)}(F_1, \mathscr{H}om\,(F_2, F_3)).\end{aligned}$$

We skip the proof.

8.4 Grothendieck topologies

We shall axiomatize the classical notion of a covering in a topological space.

Let X be a presite. Recall that all along this book, we assume (8.3.4), that is:

(8.4.1) the presite X admits products of two objects and fiber products.

Recall Notation 8.3.6.

In the sequel, we shall often write $\mathscr{S} \subset \mathscr{C}_U$ instead of $\mathscr{S} \subset \mathrm{Ob}(\mathscr{C}_U)$. We shall also often write $V \in \mathscr{C}_U$ instead of $(V \to U) \in \mathscr{C}_U$. For $\mathscr{S} \subset \mathscr{C}_U$ and $V \in \mathscr{C}_U$, we set

$$V \times_U \mathscr{S} := \{V \times_U W; W \in \mathscr{S}\},$$

a subset of $\mathscr{C}_V$.

For $\mathscr{S}_1 \subset \mathscr{C}_U$ and $\mathscr{S}_2 \subset \mathscr{C}_U$, we set

$$\mathscr{S}_1 \times_U \mathscr{S}_2 := \{V_1 \times_U V_2; V_1 \in \mathscr{S}_1, V_2 \in \mathscr{S}_2\},$$

a subset of $\mathscr{C}_U$.

For a morphism of presites $f\colon X \to Y$ (that is, a functor $f^t\colon \mathscr{C}_Y \to \mathscr{C}_X$) and $V \in \mathscr{C}_Y$, $\mathscr{S} \subset \mathscr{C}_V$, we set

$$f^t(\mathscr{S}) := \{f^t(W); W \in \mathscr{S}\},$$

a subset of $\mathscr{C}_{f^t(V)}$.

Definition 8.4.1. Let $U \in \mathscr{C}_X$. Consider two subsets $\mathscr{S}_1$ and $\mathscr{S}_2$ of $\mathrm{Ob}(\mathscr{C}_U)$. One says that $\mathscr{S}_1$ is a refinement of $\mathscr{S}_2$ if for any $U_1 \in \mathscr{S}_1$ there exists $U_2 \in \mathscr{S}_2$ and a morphism $U_1 \to U_2$ in $\mathscr{C}_U$. In such a case, we write $\mathscr{S}_1 \preceq \mathscr{S}_2$.

Remark 8.4.2. Instead of considering a subset $\mathscr{S}$ of $\mathrm{Ob}(\mathscr{C}_U)$, one may also consider a family $\mathscr{U} = \{U_i\}_{i \in I}$ of objects of $\mathscr{C}_U$ indexed by a set I. To such a family one may associate $\mathscr{S} = \mathrm{Im}\,\mathscr{U} \subset \mathrm{Ob}(\mathscr{C}_U)$. Then for $\mathscr{U}_1 = \{U_i\}_{i \in I}$ and $\mathscr{U}_2 = \{V_j\}_{j \in J}$, we say that $\mathscr{U}_1$ is a refinement of $\mathscr{U}_2$ and write $\mathscr{U}_1 \preceq \mathscr{U}_2$ if for any $i \in I$ there exists $j \in J$ and a morphism $U_i \to V_j$ in $\mathscr{C}_U$. This is equivalent to saying that $\mathrm{Im}\,\mathscr{U}_1 \preceq \mathrm{Im}\,\mathscr{U}_2$.

Of course, if the map $I \to \mathrm{Ob}(\mathscr{C}_U)$, $i \mapsto U_i$ is injective, it is equivalent to work with $\mathscr{U} = \{U_i\}_{i \in I}$ or with $\mathscr{S} = \mathrm{Im}(\mathscr{U})$.

Definition 8.4.3. Let X be a presite satisfying hypothesis (8.3.4). A Grothendieck topology (or simply “a topology”) on X is the data for each $U \in \mathscr{C}_X$ of a family $\mathrm{Cov}(U)$ of subsets of $\mathrm{Ob}(\mathscr{C}_U)$ satisfying the axioms COV1–COV4 below.

COV1 $\{U\}$ belongs to $\mathrm{Cov}(U)$.

COV2 If $\mathscr{S}_1 \in \mathrm{Cov}(U)$ is a refinement of $\mathscr{S}_2 \subset \mathrm{Ob}(\mathscr{C}_U)$, then $\mathscr{S}_2 \in \mathrm{Cov}(U)$.

COV3 If $\mathscr{S}$ belongs to $\mathrm{Cov}(U)$, then $\mathscr{S} \times_U V$ belongs to $\mathrm{Cov}(V)$ for any $(V \to U) \in \mathscr{C}_U$.

COV4 If $\mathscr{S}_1$ belongs to $\mathrm{Cov}(U)$, $\mathscr{S}_2 \subset \mathscr{C}_U$, and $\mathscr{S}_2 \times_U V$ belongs to $\mathrm{Cov}(V)$ for any $V \in \mathscr{S}_1$, then $\mathscr{S}_2$ belongs to $\mathrm{Cov}(U)$.

An element of $\mathrm{Cov}(U)$ is called a covering of U.

Intuitively, COV3 means that a covering of an open set U induces a covering on any open subset $V \subset U$, and COV4 means that if a family of open subsets of U induces a covering on each subset of a covering of U, then this family is a covering of U.

Since the category $\mathscr{C}_X$ does not necessarily admit a terminal object, the following definition is useful.

Definition 8.4.4. Let X be a presite endowed with a Grothendieck topology. A covering of X is a subset $\mathscr{S}$ of $\mathrm{Ob}(\mathscr{C}_X)$ such that $\mathscr{S} \times_X U$ belongs to $\mathrm{Cov}(U)$ for any $U \in \mathscr{C}_X$.

Definition 8.4.5. (i) A site X is a presite X satisfying hypothesis (8.3.4) and endowed with a Grothendieck topology.

(ii) A morphism of sites $f\colon X \to Y$ is a morphism of presites such that

(a) $f^t\colon \mathscr{C}_Y \to \mathscr{C}_X$ commutes with products and fiber products,

(b) for any $V \in \mathscr{C}_Y$ and $\mathscr{S} \in \mathrm{Cov}(V)$, $f^t(\mathscr{S}) \in \mathrm{Cov}(f^t(V))$.

Examples 8.4.6. a_X (i) The classical notion of a covering on a topological space X is as follows. A family $\mathscr{S} \subset \mathrm{Op}_U$ is a covering if $\bigcup_{V\in\mathscr{S}} V = U$. Axioms COV1–COV4 are clearly satisfied, and we still denote by X the site so obtained. If $f\colon X \to Y$ is a continuous map of topological spaces, it defines a morphism of sites.
(ii) Let X be a presite. The initial topology on X is defined as follows. Any subset of $\mathrm{Ob}(\mathscr{C}_U)$ is a covering. We shall denote by X_{ini} this site. Note that if X is a site, the morphism of presites $\mathrm{id}_X\colon X \to X$ induces a morphism of sites $X_{\mathrm{ini}} \to X$.
(iii) Let X be a presite. The final topology on X is defined as follows. A family $\mathscr{S} \subset \mathrm{Ob}(\mathscr{C}_U)$ is a covering of U if and only if $\{U\} \in \mathscr{S}$. Note that if X is a site, the morphism of presites $\mathrm{id}_X\colon X \to X$ induces a morphism of sites $X \to X_{\mathrm{fin}}$.
(iv) We shall denote by $\{\mathrm{pt}\}$ the set with one element and we denote this element by pt. We endow $\{\mathrm{pt}\}$ with the discrete topology. Hence, the category $\mathscr{C}_{\{\mathrm{pt}\}}$ associated with the presite $\{\mathrm{pt}\}$ has two objects, $\varnothing$ and pt and $\{\mathrm{pt}\}$ is a site. The Grothendieck topology so defined is the final topology. If X is a toplogical space, we shall usually denote by $a_X\colon X \to \{\mathrm{pt}\}$ the unique continuous map from X to $\{\mathrm{pt}\}$.
(v) Let Pt be the category with one object (let us say c) and one morphism, id_c. Then the initial and final topology on Pt differs. The empty covering is a covering of c for the initial topology, not for the final one. In the sequel, we endow Pt with the final topology. If X is a site with a terminal object X, there is a natural morphism of sites $X \to \mathrm{Pt}$, which associates the object $X \in \mathscr{C}_X$ to $c \in \mathrm{Pt}$.

(vi) Let X be a topological space. Let us endow Op_X with the following Grothendieck topology: $\mathscr{S} \subset \mathrm{Op}_U$ is a covering of U if there exists a finite subset $\mathscr{S}' \subset \mathscr{S}$ such that $\bigcup_{V\in\mathscr{S}'} V = U$. Axioms COV1–COV4 are clearly satisfied. We denote by X_{finite} the site so obtained.
(vii) Let X be a real analytic manifold. The subanalytic site X_{sa} is defined in [KS01] as follows: the objects of $\mathscr{C}_{X_{\mathrm{sa}}}$ are the relatively compact subanalytic open subsets of X and the topology is that of X_{finite}, that is, a covering of $U \in \mathscr{C}_{X_{\mathrm{sa}}}$ is a covering of U in X_{finite}. We shall develop this point in Chapter 12.
(viii) Let X be a topological space endowed with an equivalence relation $\sim$. Let $\mathscr{C}_X$ be the category of saturated open subsets (U is saturated if $x \in U$ and $x \sim y$ implies $y \in U$). We endow $\mathscr{C}_X$ with the induced topology, that is, the coverings of $U \in \mathscr{C}_X$ are the saturated coverings of U in X.
(ix) Let $\mathscr{V}$ be a universe with $\mathscr{V} \in \mathscr{U}$. Denote by $C^\infty_{\mathscr{V}}$ be the small $\mathscr{U}$-category whose objects are the real manifolds of class C^∞ belonging to $\mathscr{V}$ and morphisms are morphisms of such manifolds. Let $X \in C^\infty_{\mathscr{V}}$ and define the category $\mathscr{C}_X$ as follows. An object of $\mathscr{C}_X$ is an étale morphism $f\colon Y \to X$ in $C^\infty_{\mathscr{V}}$. (Recall that a morphism $f\colon Y \to X$ is étale if f is open and, locally on Y, f is an isomorphism onto its image.) A morphism $u\colon (Y_1 \xrightarrow{f_1} X) \to (Y_2 \xrightarrow{f_2} X)$ is a morphism $g\colon Y_1 \to Y_2$ such that $f_2 \circ g = f_1$. Necessarily, g is étale. Let us denote by X_{et} the presite so defined. We endowed it with the following topology: a family of morphism $\{U_i \xrightarrow{f_i} U\}_i$ is a covering of $U \in \mathscr{C}_X$ if U is the union of the $f_i(U_i)$'s.

Let X be a site and let $U \in \mathscr{C}_X$. To U is associated the category $\mathscr{C}_U$.

- Denoting again by U the presite associated with $\mathscr{C}_U$, the presite U satisfies (8.3.4). Moreover, the presite U admits a terminal object, namely U (i.e., $\mathrm{id}_U\colon U \to U$.)
- The functor $\mathrm{j}_U^t\colon \mathscr{C}_U \to \mathscr{C}_X$ given by $\mathrm{j}_U^t(V \to U) = V$ defines a morphism of presites:

  $$\mathrm{j}_U\colon X \to U. \tag{8.4.2}$$

- The functor $i_U^t\colon \mathscr{C}_X \to \mathscr{C}_U$ given by $i_U^t(V) = U \times_X V \to U$ defines a morphism of presites

  $$i_U\colon U \to X. \tag{8.4.3}$$

Definition 8.4.7. The induced topology by X on the presite U is defined as follows. Let $(V \to U) \in \mathscr{C}_U$. A subset $\mathscr{S} \subset \mathscr{C}_V$ is a covering of $(V \to U)$ if it is a covering of V in $\mathscr{C}_X$.

Clearly this family satisfies the axioms COV1–COV4, and thus defines a topology on the presite U.

Lemma 8.4.8. *The morphisms of presites* (8.4.2) *and* (8.4.3) *are morphisms of sites.*

The obvious verifications are left to the reader.

Example 8.4.9. Let X be a topological space, U an open subset. Note that both Op_X and Op_U admit finite projective limits, but in general j_U does not commute with such limits since it does not send the terminal object U of Op_U to the terminal object X of Op_X.

8.5 Sheaves

from now on and until the end of this book, we shall assume that the category $\mathscr{A}$ satisfies hypotheses 8.5.1 below which are satisfied in particular when choosing $\mathscr{A} = \mathbf{Set}$ or $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. See [KS06, (17.4.1) and Prop. 3.1.11].

$$\begin{cases}\text{(a) } \mathscr{A} \text{ admits small limits and small colimits and small filtered} \\ \qquad \text{colimits are exact } (\textit{i.e.} \text{ commute with finite limits}). \\ \text{(b) There exist a set } I \text{ and a functor } \lambda\colon \mathscr{A} \to \mathbf{Set}^I \text{ such that} \\ \qquad \text{(i) } \lambda \text{ commutes with small products,} \\ \qquad \text{(ii) } \lambda \text{ commutes with small filtered colimits,} \\ \qquad \text{(iii) } \lambda \text{ is conservative.}\end{cases} \tag{8.5.1}$$

Let $\mathscr{S} \subset \mathscr{C}_U$. Recall that an object $V \in \mathscr{S}$ is a morphism $V \to U$. Let $F \in \mathrm{PSh}(X, \mathscr{A})$. One defines $F(\mathscr{S})$ by the exact sequence (i.e., $F(\mathscr{S})$ is the kernel of the double arrow):

$$F(\mathscr{S}) \to \prod_{V \in \mathscr{S}} F(V) \rightrightarrows \prod_{V', V'' \in \mathscr{S}} F(V' \times_U V''). \tag{8.5.2}$$

Here the two arrows are associated with $\prod_{V \in \mathscr{S}} F(V) \to F(V') \to F(V' \times_X V'')$ and $\prod_{V \in \mathscr{S}} F(V) \to F(V'') \to F(V' \times_X V'')$.

Assume that $\mathscr{S}$ is stable by fiber products, that is, if $V \to U$ and $W \to U$ belong to $\mathscr{S}$ then $V \times_U W \to U$ belongs to $\mathscr{S}$. In this case, looking at $\mathscr{S}$ as a full subcategory of $\mathscr{C}_U$, the natural morphism $\varprojlim_{(V \to U) \in \mathscr{S}} F(V) \to \prod_{V \in \mathscr{S}} F(V)$ factorizes through $F(\mathscr{S})$ and one checks that

$$\varprojlim_{(V \to U) \in \mathscr{S}} F(V) \xrightarrow{\sim} F(\mathscr{S}). \tag{8.5.3}$$

Note that, if $\mathscr{A} = \mathbf{Set}$, a section $s \in F(\mathscr{S})$ is the data of a family of sections $\{s_V \in F(V)\}_{V \in \mathscr{S}}$ such that for any $V', V'' \in \mathscr{S}$,

$$s_{V'}|_{V' \times_X V''} = s_{V''}|_{V' \times_X V''}.$$

Here, $s_{V'}|_{V' \times_X V''}$ is the image of s by the restriction map associated with the natural morphism $V' \times_X V'' \to V'$ and similarly with V''.

For a presheaf F, there is a natural map

$$F(U) \to F(\mathscr{S}). \tag{8.5.4}$$

Definition 8.5.1. (i) One says that a presheaf F is separated if for any $U \in \mathscr{C}_X$ and any covering $\mathscr{S}$ of U, the natural morphism $F(U) \to F(\mathscr{S})$ is a monomorphism.

(ii) One says that a presheaf F is a sheaf if for any $U \in \mathscr{C}_X$ and any covering $\mathscr{S}$ of U, the natural map $F(U) \to F(\mathscr{S})$ is an isomorphism.

(iii) One denotes by $\mathrm{Sh}(X,\mathscr{A})$ the full subcategory of $\mathrm{PSh}(X,\mathscr{A})$ whose objects are sheaves and by $\iota_X\colon \mathrm{Sh}(X,\mathscr{A})\to \mathrm{PSh}(X,\mathscr{A})$ the forgetful functor. If there is no risk of confusion, we write ι instead of ι_X, or even, we do not write ι.

(iv) One sets $\mathrm{Sh}(X)=\mathrm{Sh}(X,\mathbf{Set})$ and $\mathrm{Mod}(\mathbf{k}_X)=\mathrm{Sh}(X,\mathrm{Mod}(\mathbf{k}))$. One calls an object of $\mathrm{Mod}(\mathbf{k}_X)$ a $\mathbf{k}$-abelian sheaf, or an abelian sheaf, for short.

Remark 8.5.2. To check that a presheaf F is a sheaf, it is enough to verify that the map $F(U)\to F(\mathscr{S})$ is an isomorphism for any U and any covering $\mathscr{S}$ stable by fiber products.

Remark 8.5.3. Assume that $\mathscr{A}$ is either the category **Set** or the category $\mathrm{Mod}(\mathbf{k})$. Let F be a presheaf on X and consider the two conditions below.

S1 For any $U\in\mathscr{C}_X$, any covering $\mathscr{S}$ of U, any $s,t\in F(U)$ satisfying $s|_V=t|_V$ for all $V\in\mathscr{S}$, one has $s=t$.

S2 For any $U\in\mathscr{C}_X$, any covering $\mathscr{S}$ of U, any family $\{s_V\in F(V)\}_{V\in\mathscr{S}}$ satisfying $s_V|_{V\times_U W}=s_W|_{V\times_U W}$ for all $U,V\in\mathscr{S}$, there exists $s\in F(U)$ with $s|_V=s_V$ for all $V\in\mathscr{S}$.

Then a presheaf F is separated (resp. is a sheaf) if and only if it satisfies S1 (resp. S1 and S2).

Assume that X is a topological space and $\mathscr{A}=\mathrm{Mod}(\mathbf{k})$. Let $F\in\mathrm{Mod}(\mathbf{k}_X)$.

- Let $\{U_i\}_{i\in I}$ be a family of disjoint open subsets. Then $F(\bigsqcup_i U_i)=\prod_i F(U_i)$. In particular, $F(\varnothing)=0$.
- One defines the support of F, denoted $\operatorname{supp} F$, as the complementary of the union of all open subsets U of X such that $F|_U=0$. Note that $F|_{X\setminus\operatorname{supp} F}=0$.
- Let $s\in F(U)$. One defines its support, denoted by $\operatorname{supp} s$, as the complementary of the union of all open subsets U of X such that $s|_U=0$.

Of course, these notions of support have no meaning on a site.

Theorem 8.5.4. *Let $F\in\mathrm{PSh}(X,\mathscr{A})$ and $G\in\mathrm{Sh}(X,\mathscr{A})$. The presheaf $\mathscr{H}om\,(F,\iota_X G)$ is a sheaf of sets on X. (A sheaf of $\mathbf{k}$-modules in case $\mathscr{A}=\mathrm{Mod}(\mathbf{k})$.)*

In the sequel, we shall not write ι_X.

Proof. Let $U\in\mathscr{C}_X$ and let $\mathscr{S}$ be a covering of U. We shall check conditions S1 and S2 as in Remark 8.5.3. Consider the diagram

$$\begin{tikzcd}
F(U) \arrow[r] \arrow[d] & F(\mathscr{S}) \arrow[r] \arrow[d] & \prod_{V\in\mathscr{S}} F(V) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{V',V''\in\mathscr{S}} F(V'\times_U V'') \arrow[d] \\
G(U) \arrow[r, "\sim"] & G(\mathscr{S}) \arrow[r] & \prod_{V\in\mathscr{S}} G(V) \arrow[r, shift left] \arrow[r, shift right] & \prod_{V',V''\in\mathscr{S}} G(V'\times_U V'')
\end{tikzcd}$$

(S1) Let $\varphi,\psi\colon F|_U\to G|_U$ be two morphisms defined on U. Denote by φ_V,ψ_V their restriction to $V\in\mathscr{S}$. These families of morphisms define the morphisms

$\varphi_{\mathscr{S}}, \psi_{\mathscr{S}} \colon F(\mathscr{S}) \to G(\mathscr{S})$. Assuming that $\varphi_V = \psi_V$ for all V, we get $\varphi_{\mathscr{S}} = \psi_{\mathscr{S}}$ hence $\varphi(U) = \psi(U)$ and by the same argument, $\varphi(V) = \psi(V)$ for any $V \to U$.

(S2) Let $\{\varphi_V\}_V$ be a family of morphisms $\varphi_V \colon F|_V \to G|_V$ and assume that $\varphi_V = \varphi_W$ on $V \times_U W$. Then this family of morphisms defines a morphism $\varphi_{\mathscr{S}} \colon F(\mathscr{S}) \to G(\mathscr{S})$. One constructs $\varphi(U)$ as the composition $F(U) \to F(\mathscr{S}) \xrightarrow{\varphi_{\mathscr{S}}} G(\mathscr{S}) \xleftarrow{\sim} G(U)$. Replacing U with $V \to U$, one checks easily that the family of morphisms $\{\varphi(V)\}_{V \to U}$ so constructed defines a morphism of presheaves $F|_U \to G|_U$. ☐

We shall still denote by $\mathscr{H}om\,(F, G)$ the sheaf given by Theorem 8.5.4.

Corollary 8.5.5. *Let* $\varphi \colon F \to G$ *be a morphism in* $\mathrm{Sh}(X, \mathscr{A})$*. Assume that there is a covering* $\mathscr{S}$ *of* X *such that* $\varphi_V \colon F|_V \to G|_V$ *is an isomorphism for any* $V \in \mathscr{S}$*. Then* φ *is an isomorphism.*

Proof. For $V \in \mathscr{S}$, denote by ψ_V the inverse of φ_V. Then for any $V, W \in \mathscr{S}$, $\psi_V|_{V\times_X W} = \psi_W|_{V \times_X W}$. By Theorem 8.5.4, there exists $\psi \colon G \to F$ such that $\psi|_V = \psi_V$ for all $V \in \mathscr{S}$. Clearly $\psi \circ \varphi = \mathrm{id}_F$ and $\varphi \circ \psi = \mathrm{id}_G$. ☐

In other words, a morphism of sheaves which is locally an isomorphism, is an isomorphism. In § 8.8 we shall construct sheaves which are locally isomorphic without being isomorphic.

Example 8.5.6. We have already encountered many classical examples of sheaves on a topological space or on a (real or complex) manifold in Example 8.1.6.

8.6 Sheaf associated with a presheaf

Recall that we have made Hypotheses 8.3.1.

In this section, we shall explain how to construct the "sheaf associated with a presheaf". More precisely, we shall show that the natural forgetful functor ι_X : $\mathrm{Sh}(X, \mathscr{A}) \to \mathrm{PSh}(X, \mathscr{A})$ which, to a sheaf F, associates the underlying presheaf, admits a left adjoint.

Let $U \in \mathscr{C}_X$ and let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two subsets of $\mathscr{C}_U$. Notice first that the relation $\mathscr{S}_1 \preceq \mathscr{S}_2$ is a pre-order on $\mathrm{Cov}(U)$. Hence, $\mathrm{Cov}(U)$ inherits a structure of a category:

$$\mathrm{Hom}_{\,\mathrm{Cov}(U)}(\mathscr{S}_1, \mathscr{S}_2) = \begin{cases} \{\mathrm{pt}\} & \text{if } \mathscr{S}_1 \text{ is a refinement of } \mathscr{S}_2, \\ \varnothing & \text{otherwise.} \end{cases}$$

For $\mathscr{S}_1, \mathscr{S}_2 \in \mathrm{Cov}(U)$, $\mathscr{S}_1 \times_U \mathscr{S}_2$ again belongs to $\mathrm{Cov}(U)$. Therefore:

Lemma 8.6.1. *The category* $\mathrm{Cov}(U)$ *is cofiltered* (*i.e., the opposite category is filtered*)*.*

Lemma 8.6.2. *Let* $F \in \mathrm{PSh}(X, \mathscr{A})$ *and let* $U \in \mathscr{C}_X$*. Then* F *naturally defines a functor* $\mathrm{Cov}(U)^{\mathrm{op}} \to \mathscr{A}$*.*

Proof. Let $\mathscr{S}_1 \preceq \mathscr{S}_2$. We shall construct a natural morphism $F(\mathscr{S}_2) \to F(\mathscr{S}_1)$. For $V_1 \in \mathscr{S}_1$ we construct $F(\mathscr{S}_2) \to F(V_1)$ by choosing $V_2 \in \mathscr{S}_2$ and a morphism $V_1 \to V_2$. The composition $F(\mathscr{S}_2) \to \prod_{V \in \mathscr{S}_2} F(V) \to F(V_2) \to F(V_1)$ does not depend on the choice of $V_1 \to V_2$. Indeed, if we have two morphisms $V_1 \to V_2'$ and

$V_1 \to V_2''$, these morphisms factorize through $V_1 \to V_2' \times_{V_1} V_2''$ and the composition $F(\mathscr{S}_2) \to F(V_2') \to F(V_2' \times_{V_1} V_2'') \to F(V_1)$ is the same as the composition $F(\mathscr{S}_2) \to F(V_2'') \to F(V_2' \times_{V_1} V_2'') \to F(V_1)$.

The family of morphisms $F(\mathscr{S}_2) \to F(V_1)$, $V_1 \in \mathscr{S}_1$, defines $F(\mathscr{S}_2) \to F(\mathscr{S}_1)$ and one checks easily the functoriality of this construction. □

One defines the presheaf F^+ by setting for all $U \in \mathscr{C}_X$:

$$F^+(U) = \operatorname*{colim}_{\mathscr{S} \in \mathrm{Cov}(U)^{\mathrm{op}}} F(\mathscr{S}). \tag{8.6.1}$$

For any $V \to U$, the morphism $F^+(U) \to F^+(V)$ is defined by the sequence of morphisms

$$F^+(U) = \operatorname*{colim}_{\mathscr{S} \in \mathrm{Cov}(U)^{\mathrm{op}}} F(\mathscr{S}) \to \operatorname*{colim}_{\mathscr{S} \in \mathrm{Cov}(U)^{\mathrm{op}}} F(V \times_U \mathscr{S}) \to \operatorname*{colim}_{\mathscr{T} \in \mathrm{Cov}(V)^{\mathrm{op}}} F(\mathscr{T}) = F^+(V).$$

The second arrow is well-defined since $V \times_U \mathscr{S} \in \mathrm{Cov}(V)$.

Hence, $F \mapsto F^+$ defines a functor $^+ : \mathrm{PSh}(X, \mathscr{A}) \to \mathrm{PSh}(X, \mathscr{A})$. Moreover for each $U \in \mathscr{C}_X$, the maps $F(U) \to F(\mathscr{S}), \mathscr{S} \in \mathrm{Cov}(U)$ define $F(U) \to \operatorname*{colim}_{\mathscr{S} \in \mathrm{Cov}(U)^{\mathrm{op}}} F(\mathscr{S}) = F^+(U)$. Hence, there is a morphism of functors $\alpha : \mathrm{id} \to {}^+$.

Theorem 8.6.3. *Assume that $\mathscr{A}$ satisfies (8.5.1).*

(i) *If F is a separated presheaf, then $F \to F^+$ is a monomorphism.*

(ii) *If F is a sheaf, then $F \to F^+$ is an isomorphism.*

(iii) *For any presheaf F, F^+ is a separated presheaf.*

(iv) *For any separated presheaf F, F^+ is a sheaf.*

(v) *The functor $^a := {}^{++} \colon \mathrm{PSh}(X, \mathscr{A}) \to \mathrm{Sh}(X, \mathscr{A})$ is a left adjoint to the embedding functor $\iota_X \colon \mathrm{Sh}(X, \mathscr{A}) \to \mathrm{PSh}(X, \mathscr{A})$.*

(vi) *The functor $^+ \colon \mathrm{PSh}(X, \mathscr{A}) \to \mathrm{PSh}(X, \mathscr{A})$ is left exact and the functor ^{a} is exact.*

Proof. In the course of the proof of (iii) and (iv) we shall use Hypothesis (8.5.1) (ii) and reduce to the case where $\mathscr{A} = \mathbf{Set}^I$.

(i) By the hypothesis, for any open set U and any covering $\mathscr{S}$ of U, the morphism $F(U) \to F(\mathscr{S})$ is a monomorphism. Since $\mathrm{Cov}(U)$ is cofiltered, $F(U) \to \operatorname*{colim}_{\mathscr{S} \in \mathrm{Cov}(U)} F(\mathscr{S}) = F^+(U)$ is a monomorphism.

(ii) By the hypothesis, for any open set U and any covering $\mathscr{S}$ of U, the morphism $F(U) \to F(\mathscr{S})$ is an isomorphism. The result follows.

(iii) Let $s_1, s_2 \in F^+(U)$ and let $\mathscr{S}_0 \in \mathrm{Cov}(U)$ be such that $s_1 = s_2$ in $F^+(\mathscr{S}_0)$ is 0. We may represent s_1 and s_2 by sections $s_1, s_2 \in F(\mathscr{S})$ where $\mathscr{S} \in \mathrm{Cov}(U)$. Then the images of s_1 and s_2 by the composition of the maps $F(\mathscr{S}) \to F^+(U) \to F^+(\mathscr{S}_0)$ coincide. For each $(V \to U) \in \mathscr{S}_0$ there exists $\mathscr{S}_V \in \mathrm{Cov}(V)$ such that the image of s_1 and s_2 in $F(\mathscr{S}_V \times_U \mathscr{S})$ coincide. By Axiom Cov 4, we get a covering of U on which $s_1 = s_2$, hence $s_1 = s_2$ in $F^+(U)$.

(iv) Let us prove that the map $F^+(U) \to F^+(\mathscr{S}_0)$ is an epimorphism. Recall that $F^+(\mathscr{S}_0)$ is the kernel of $\prod_{V\in\mathscr{S}_0} F^+(V) \rightrightarrows \prod_{V',V''\in\mathscr{S}_0} F^+(V'\times_U V'')$. One may represent a section of $F^+(\mathscr{S}_0)$ (with $\mathscr{S}_V$ a covering of $V \in \mathscr{S}_0$) by a section s of $\prod_{V\in\mathscr{S}_0} F(\mathscr{S}_V)$ whose two images in $\prod_{V',V''\in\mathscr{S}_0} F^+(\mathscr{S}_{V'}\times_U \mathscr{S}_{V''})$ coincide. Since F is separated, these two images already coincide in $\prod_{V',V''\in\mathscr{S}_0} F(\mathscr{S}_{V'}\times_U \mathscr{S}_{V''})$. Therefore, s belongs to $\ker\big(\prod_{V\in\mathscr{S}_0} F(\mathscr{S}_V) \rightrightarrows \prod_{V',V''\in\mathscr{S}_0} F(\mathscr{S}_{V'}\times_U \mathscr{S}_{V''})\big)$. Again by Axiom Cov 4, we get a covering $\mathscr{S}_1$ of U such that $s \in F(\mathscr{S}_1)$.

(v) Let $G \in \mathrm{Sh}(X,\mathscr{A})$. The morphism $F \to F^+$ defines the morphism

$$\lambda\colon \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F^+,G) \to \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F,G)$$

and the functor ${}^+$ defines the morphism

$$\begin{aligned}\operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F,G) &\to \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F^+,G^+)\\ &\simeq \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F^+,G).\end{aligned}$$

One checks that these two morphisms are inverse one to each other. Therefore λ is an isomorphism. Replacing F with F^+, the result follows.

(vi) To prove that ${}^+$ is left exact, it is enough to check that for each $U \in \mathscr{C}_X$, the functor $F \to F^+(U)$ is left exact. Since $F^+(U) = \operatorname{colim} F(\mathscr{S})$, where $\mathscr{S}$ ranges over the cofiltered category $\mathrm{Cov}(U)$, it is enough to check that the functor $F \to F(\mathscr{S})$ is left exact. This follows from (8.5.2) and the fact that $F \mapsto F(V)$ is exact.

Since ${}^a = {}^{++}$, this functor is left exact. Since it is a left adjoint, it is right exact. $\square$

In the sequel, we shall often omit to write the symbol ι_X. Hence, (v) may be written as follows with $F \in \mathrm{PSh}(X,\mathscr{A})$ and $G \in \mathrm{Sh}(X,\mathscr{A})$

$$\operatorname{Hom}_{\mathrm{Sh}(X,\mathscr{A})}(F^a,G) \xrightarrow{\sim} \operatorname{Hom}_{\mathrm{PSh}(X,\mathscr{A})}(F,G). \tag{8.6.2}$$

Definition 8.6.4. (i) For F a presheaf on X, the sheaf F^a is called the sheaf associated with F.

(ii) We denote by $\theta\colon \mathrm{id} \to \iota_X \circ {}^a$ the natural morphism of functor associated with the pair of adjoint functor $({}^a, \iota_X)$.

Hence, Theorem 8.6.3 (vi) may be formulated as follows: any morphism of presheaves $\varphi\colon F \to G$, with G a sheaf, factorizes uniquely as

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\theta"'] & G. \\ F^a \arrow[ru] & \end{tikzcd} \tag{8.6.3}$$

Remark 8.6.5. When X is a topological space and $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, the construction of the sheaf F^a is much easier. Define:

$$\begin{aligned}F^a(U) = \{s : U \to \bigsqcup_{x\in U} F_x\ ;\ s(x) \in F_x \text{ and for all } x \in U,\\ \text{there exists an open neighborhood } V \text{ of } x \text{ in } U\\ \text{and } t \in F(V) \text{ with } t_y = s(y) \text{ for all } y \in V\}.\end{aligned}$$

Define $\theta : F \to F^a$ as follows. To $s \in F(U)$, one associates the section of F^a:

$$(x \mapsto s_x) \in F^a(U).$$

One checks that (F^a, θ) has the required properties, that is, any morphism of presheaves $\varphi\colon F \to G$ factorizes uniquely as in (8.6.3). Details are left to the reader.

Theorem 8.6.6. (i) *The category* $\mathrm{Sh}(X, \mathscr{A})$ *admits small limits and such limits commute with the functor* ι_X.

(ii) *The category* $\mathrm{Sh}(X, \mathscr{A})$ *admits small colimits. More precisely, if* $\{F_i\}_{i\in I}$ *is an inductive system of sheaves, its colimit is the sheaf associated with its colimit in* $\mathrm{PSh}(\mathbf{k}_X)$.

(iii) *The functor* $\iota_X\colon \mathrm{Sh}(X, \mathscr{A}) \to \mathrm{PSh}(X, \mathscr{A})$ *is fully faithful and commutes with small limits (in particular, it is left exact). The functor* ${}^a : \mathrm{PSh}(X, \mathscr{A}) \to \mathrm{Sh}(X, \mathscr{A})$ *commutes with small colimits and is exact.*

(iv) *Small filtered colimits are exact in* $\mathrm{Sh}(X, \mathscr{A})$.

Proof. (i) Let $\{F_i\}_{i\in I}$ be a small projective system of sheaves, let $U \in \mathscr{C}_X$ and let $\mathscr{S} \in \mathrm{Cov}(U)$. By the definition of $F(\mathscr{S})$, one sees that the morphism $F(U) \to F(\mathscr{S})$ commutes with limits, that is, $(\varprojlim_i F_i)(U) \xrightarrow{\sim} (\varprojlim_i F_i)(\mathscr{S})$. Hence a projective limit of sheaves in the category $\mathrm{PSh}(X, \mathscr{A})$ is a sheaf. The fact that this sheaf is a projective limit in $\mathrm{Sh}(X, \mathscr{A})$ of the projective system $\{F_i\}_{i\in I}$ follows from the fact that the forgetful functor $\mathrm{PSh}(X, \mathscr{A}) \to \mathrm{Sh}(X, \mathscr{A})$ is fully faithful:

$$\begin{aligned}\mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(G, \varprojlim_i F_i) &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(G, \varprojlim_i F_i)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(G, F_i)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(G, F_i).\end{aligned}$$

(ii) Let $\{F_i\}_{i\in I}$ is a small inductive system of sheaves. Let us denote by $\text{“}\varinjlim_i\text{”} F_i$ its inductive limit in the category $\mathrm{PSh}(X.\mathscr{A})$ and let $G \in \mathrm{Sh}(X, \mathscr{A})$. We have the chain of isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}((\text{“}\varinjlim_i\text{”} F_i)^a, G) &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(\text{“}\varinjlim_i\text{”} F_i, G)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(F_i, G)\\ &\simeq \varprojlim_i \mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(F_i, G).\end{aligned}$$

(iii) The functor ι_X is fully faithful by definition. By adjunction, ι_X commutes with small limits and ^{a} commutes with small colimits. It is exact by Theorem 8.6.3.

(iv) Small filtered colimits are exact in the category $\mathscr{A}$, whence in the category $\mathrm{PSh}(X, \mathscr{A})$. Then the result follows since ^{a} is exact. ☐

The functor $\Gamma(U;\bullet)$

We have already introduced the functors $\Gamma(U;\bullet)$ and $\Gamma(X;\bullet)$ on presheaves in (8.2.2) and (8.2.3). We keep the same notation for sheaves.

Proposition 8.6.7. *The functors $\Gamma(U;\bullet)\colon \mathrm{Sh}(X,\mathscr{A})\to\mathscr{A}$ is left exact. The same result holds for $\Gamma(X;\bullet)$.*

Proof. The functor ι_X is left exact and the functor $F\mapsto F(U)$ is exact on $\mathrm{PSh}(X,\mathscr{A})$. ☐

Remark 8.6.8. As usual, one endows the set pt is with its natural topology (consisting of two open sets, pt and $\varnothing$). Then the functor

$$\Gamma(\mathrm{pt};\bullet)\colon \mathrm{Sh}(X,\mathscr{A})\to\mathscr{A}$$

is an equivalence of categories. In the sequel, we shall identify these two categories.

The functor $\Gamma(X;\bullet)$ is not exact in general, as shown by the example below, a variant of Example 8.1.7 (e).

Example 8.6.9. Let X be a complex curve. The holomorphic De Rham complex reads as $0\to\mathbb{C}_X\to\mathscr{O}_X\xrightarrow{d}\Omega_X\to 0$. Applying the functor $\Gamma(U;\bullet)$ for an open subset U of X, we find the complex $0\to\mathbb{C}_X(U)\to\mathscr{O}_X(U)\xrightarrow{d}\Omega_X(U)\to 0$. Choosing for example $X=\mathbb{C}$ and $U=\mathbb{C}\setminus\{0\}$, this complex is no more exact.

8.7 Operations on sheaves

Direct and inverse images of sheaves

Let $f\colon X\to Y$ be a morphism of sites. Recall Definitions 8.3.1 and 8.4.5.

Proposition 8.7.1. *Let $F\in\mathrm{Sh}(X,\mathscr{A})$. Then the presheaf f_*F is a sheaf on Y.*

Proof. Let $V\in\mathscr{C}_Y$ and let $\mathscr{S}$ be a covering of V. Since $f^t\mathscr{S}$ is a covering of f^tV, we get the chain of isomorphisms

$$f_*F(V)=F(f^t(V))\simeq F(f^t(\mathscr{S}))=f_*F(\mathscr{S}).$$

☐

Hence, the functor $f_*\colon \mathrm{PSh}(\mathbf{k}_X)\to\mathrm{PSh}(Y,\mathscr{A})$ induces a functor (we keep the same notation)

$$f_*\colon \mathrm{Sh}(X,\mathscr{A})\to\mathrm{Sh}(Y,\mathscr{A}).$$

Definition 8.7.2. Let $G\in\mathrm{Sh}(Y,\mathscr{A})$. One denotes by $f^{-1}G$ the sheaf on X associated with the presheaf $f^\dagger G$ and calls it the inverse image of G. In other words, $f^{-1}G=(f^\dagger G)^a$.

Theorem 8.7.3. *Let $f\colon X\to Y$ be a morphism of sites.*

(i) *The functor* $f^{-1}\colon \mathrm{Sh}(Y,\mathscr{A}) \to \mathrm{Sh}(X,\mathscr{A})$ *is left adjoint to* f_*. *In other words, there is an isomorphism*

$$\mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(f^{-1}G, F) \simeq \mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(G, f_*F)$$

functorial with respect to $F \in \mathrm{Sh}(X,\mathscr{A})$ *and* $G \in \mathrm{Sh}(Y,\mathscr{A})$.

(ii) *The functor* f_* *commutes with small limits. In particular, it is is left exact.*

(iii) *The functor* f^{-1} *commutes with small colimits and is exact.*

(iv) *There are natural morphisms of functors* $\mathrm{id} \to f_*f^{-1}$ *and* $f^{-1}f_* \to \mathrm{id}$.

Proof. (i) Denote for a while by "f_*" the direct image in the categories of presheaves. Since $f^\dagger$ is left adjoint to "f_*" and ^{a} is left adjoint to ι_X, $f^{-1} = {}^a \circ f^\dagger$ is left adjoint to $f_* =$ "f_*" $\circ\, \iota_X$.

(ii) and (iv) as well as the first part of (iii) follow from the adjunction property.

(iii) It remains to prove that the functor f^{-1} is left exact. The functor ^{a} being exact, it is enough to prove that $f^\dagger\colon \mathrm{Sh}(Y,\mathscr{A}) \to \mathrm{PSh}(X,\mathscr{A})$ is left exact, hence that for $U \in \mathscr{C}_X$, the functor

$$G \mapsto \operatorname*{colim}_{(U\to f^t(V))\in(\mathscr{C}_Y^U)^{\mathrm{op}}} G(V)$$

is exact. This follows from the fact that, $\mathscr{C}_Y$ admitting fiber products, the category $(\mathscr{C}_Y^U)^{\mathrm{op}}$ is either empty or is filtered. □

Corollary 8.7.4. *Let* $G \in \mathrm{PSh}(Y,\mathscr{A})$. *Then* $(f^\dagger G)^a \xrightarrow{\sim} f^{-1}(G^a)$.

Proof. One has the chain of isomorphisms, functorial with respect to $F \in \mathrm{Sh}(X,\mathscr{A})$:

$$\begin{aligned}\mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}((f^\dagger G)^a, F) &\simeq \mathrm{Hom}_{\,\mathrm{PSh}(X,\mathscr{A})}(f^\dagger G, F) \simeq \mathrm{Hom}_{\,\mathrm{PSh}(Y,\mathscr{A})}(G, f_*F)\\ &\simeq \mathrm{Hom}_{\,\mathrm{Sh}(Y,\mathscr{A})}(G^a, f_*F) \simeq \mathrm{Hom}_{\,\mathrm{Sh}(X,\mathscr{A})}(f^{-1}(G^a), F).\end{aligned}$$

Hence, the result follows from the Yoneda lemma. □

Consider two morphisms of sites $f\colon X \to Y$ and $g\colon Y \to Z$.

Proposition 8.7.5. (i) $g \circ f\colon X \to Z$ *is a morphism of sites.*

(ii) *One has natural isomorphisms of functors*

$$g_* \circ f_* \simeq (g \circ f)_*,\ f^{-1} \circ g^{-1} \simeq (g \circ f)^{-1}.$$

Proof. (i) is obvious.

(ii) The functoriality of direct images for presheaves is clear (see Proposition 8.3.4). It then follows for sheaves by Proposition 8.7.1. The functoriality of inverse images follows by adjunction. □

Examples 8.7.6. Assume that $f\colon X \to Y$ is a morphism of topological spaces.

(i) For $x \in X$,

$$(f^{-1}G)_x \simeq (f^\dagger G)_x \simeq G_{f(x)}. \tag{8.7.1}$$

(ii) Denote by $i_x\colon \{x\} \hookrightarrow X$ the embedding of $x \in X$ into X. Then, for $F \in \mathrm{Sh}(X,\mathscr{A})$

$$F_x \simeq i_x^{-1}F.$$

Examples 8.7.7. (i) Let $Z = \{a, b\}$ be a set with two elements, let Y be a topological space and let $X = Z \times Y \simeq Y \bigsqcup Y$, the disjoint union of two copies of Y. Let $f\colon X \to Y$ be the projection. Then $f_* f^{-1} G \simeq G \oplus G$. In fact, if V is open in Y, then $\Gamma(V; f_* f^{-1} G) \simeq \Gamma(V \bigsqcup V; f^{-1} G) \simeq \Gamma(V; G) \oplus \Gamma(V; G)$.

(ii) Let $X = Y = \mathbb{C} \setminus \{0\}$, and let $f : X \to Y$ be the map $z \mapsto z^2$, where z denotes a holomorphic coordinate on $\mathbb{C}$. If D is an open disk in Y, $f^{-1}D$ is isomorphic to the disjoint union of two copies of D. Hence, the sheaf $f_* \mathbf{k}_X|_D$ is isomorphic to $\mathbf{k}_D^2$, the constant sheaf of rank two on D. However, $\Gamma(Y; f_* \mathbf{k}_X) = \Gamma(X; \mathbf{k}_X) = \mathbf{k}$, which shows that the sheaf $f_* \mathbf{k}_X$ is not isomorphic to $\mathbf{k}_Y^2$.

(iii) Let $f\colon X \to Y$ be a morphism of topological spaces. To each open subset $V \subset Y$ is associated a natural "pull-back" map: $\Gamma(V; \mathscr{C}_Y^0) \to \Gamma(V; f_* \mathscr{C}_X^0)$ defined by $\varphi \mapsto \varphi \circ f$. We obtain a morphism $\mathscr{C}_Y^0 \to f_* \mathscr{C}_X^0$, hence a morphism:

$$f^{-1} \mathscr{C}_Y^0 \to \mathscr{C}_X^0.$$

For example, if X is closed in Y and f is the injection, $f^{-1} \mathscr{C}_Y^0$ will be the sheaf on X of continuous functions on Y defined in a neighborhood of X. If f is a topological submersion (locally on X, f is isomorphic to a projection $Y \times Z \to Y$), then $f^{-1} \mathscr{C}_Y^0$ will be the subsheaf of $\mathscr{C}_X^0$ consisting of functions locally constant on the fibers of f.

(iv) Let $i_S\colon S \hookrightarrow X$ be the embedding of a closed subset S of a topological space X. Then the functor i_{S*} is exact.

Restriction and extension of sheaves

Let X be a site and let $U \in \mathscr{C}_X$. We have already defined the morphisms of sites $\mathrm{j}_U\colon X \to U$ and $i_U\colon U \to X$ and we have proved in Proposition 8.3.7 the isomorphism of functors of presheaves

$$\mathrm{j}_{U*} \simeq i_U^{\dagger}, \quad \mathrm{j}_U^{\ddagger} \simeq i_{U*}. \tag{8.7.2}$$

Proposition 8.7.8. *Let $U \in \mathscr{C}_X$.*

(i) *The functor $i_U^{\dagger}$ sends $\mathrm{Sh}(X, \mathscr{A})$ to $\mathrm{Sh}(U, \mathscr{A})$ and the functor $\mathrm{j}_U^{\ddagger}$ sends $\mathrm{Sh}(U, \mathscr{A})$ to $\mathrm{Sh}(X, \mathscr{A})$.*

(ii) *The functor $\mathrm{j}_{U*}\colon \mathrm{Sh}(X, \mathscr{A}) \to \mathrm{Sh}(U, \mathscr{A})$ commutes with small limits and colimits and in particular is exact. Moreover, $\mathrm{j}_{U*} \simeq i_U^{-1}$.*

(iii) *The functor $\mathrm{j}_U^{-1}\colon \mathrm{Sh}(U, \mathscr{A}) \to \mathrm{Sh}(X, \mathscr{A})$ is exact.*

Proof. (i) follows from (8.7.2).

(ii) The functor j_{U*} admits both a right and a left adjoint. The isomorphism $\mathrm{j}_{U*} \simeq i_U^{-1}$ follows from (i).

(iii) This is a particular case of Theorem 8.7.3. □

Notation 8.7.9. One sets

$$i_{U!} := \mathrm{j}_U^{-1}, \quad \mathrm{Sh}(U, \mathscr{A}) \to \mathrm{Sh}(X, \mathscr{A}), \tag{8.7.3}$$

and for $F \in \mathrm{Sh}(X, \mathscr{A})$:

$$\begin{aligned} F|_U &:= i_U^{-1} F \simeq \mathrm{j}_{U*} F, \\ F_U &:= i_{U!} i_U^{-1} F = \mathrm{j}_U^{-1}\, i_U^{-1} F \\ \Gamma_U F &:= i_{U*} i_U^{-1} F. \end{aligned} \tag{8.7.4}$$

Hence, $i_{U!}$ is exact and Γ_U is left exact. Moreover,

$$(\Gamma_U F)(X) \simeq F(U). \tag{8.7.5}$$

We thus have two pairs of adjoint functors $(\mathrm{j}_U^{-1}, \mathrm{j}_{U*})$, $(\mathrm{j}_{U*}, \mathrm{j}_U^{\ddagger})$:

$$\begin{tikzcd} \mathrm{Sh}(U, \mathscr{A}) \arrow[r, shift left=2, "\mathrm{j}_U^{-1}"] \arrow[r, leftarrow, "\mathrm{j}_{U*}" description] \arrow[r, shift right=2, "\mathrm{j}_U^{\ddagger}"'] & \mathrm{Sh}(X, \mathscr{A}), \end{tikzcd} \tag{8.7.6}$$

which are also written as two pairs of adjoint functors $(i_{U!}, i_U^{-1})$, (i_U^{-1}, i_{U*}):

$$\begin{tikzcd} \mathrm{Sh}(U, \mathscr{A}) \arrow[r, shift left=2, "i_{U!}"] \arrow[r, leftarrow, "i_U^{-1}" description] \arrow[r, shift right=2, "i_{U*}"'] & \mathrm{Sh}(X, \mathscr{A}). \end{tikzcd} \tag{8.7.7}$$

For $V \to U$ a morphism in $\mathscr{C}_X$, there are natural morphisms :

$$F_V \to F_U \to F \to \Gamma_U F \to \Gamma_V F.$$

Also note that $((\,\bullet\,)_U, \Gamma_U(\,\bullet\,))$ is a pair of adjoint functors.

Proposition 8.7.10. *For $U, V \in \mathscr{C}_X$ there are natural isomorphisms*

$$(F_V)_U \simeq F_{U \times_X V},\ \Gamma_U(\Gamma_V F) \simeq \Gamma_{U \times_X V}(F).$$

Proof. By adjunction, it is enough to prove the second isomorphism. One has for $W \in \mathscr{C}_X$:

$$\begin{aligned} \Gamma_U(\Gamma_V(F))(W) &\simeq \Gamma_V(F)(U \times_X W) \\ &\simeq F(V \times_X U \times_X W) \simeq \Gamma_{U \times_X V}(F)(W). \end{aligned}$$

□

Let $f\colon X \to Y$ be a morphism of sites. Let $V \in \mathscr{C}_Y$, set $U = f^t(V)$ and denote by $f_V\colon U \to V$ the morphism of sites associated with the functor $f_V^t\colon \mathscr{C}_V \to \mathscr{C}_U$ deduced from f^t. We get the commutative diagram of sites

$$\begin{tikzcd} U \arrow[r, "i_U"] \arrow[d, "f_V"] & X \arrow[r, "j_U"] \arrow[d, "f"] & U \arrow[d, "f_V"] \\ V \arrow[r, "i_V"'] & Y \arrow[r, "j_V"'] & V. \end{tikzcd} \tag{8.7.8}$$

Proposition 8.7.11. *There are natural isomorphisms of functors*

$$\begin{gathered} i_V^{-1} f_* \simeq f_{V*} i_U^{-1},\ f^{-1} i_{V!} \simeq i_{U!} f_V^{-1}, \\ f_*\, \mathrm{j}_U^{\ddagger} \simeq \mathrm{j}_V^{\ddagger}\, f_{V*},\ \mathrm{j}_{U*} f^{-1} \simeq f_V^{-1} \mathrm{j}_{V*}. \end{gathered}$$

Proof. This follows from the isomorphisms $\mathrm{j}_{V*}f_* \simeq f_{V*}\mathrm{j}_{U*}$, $f^{-1}\mathrm{j}_V^{-1} \simeq \mathrm{j}_U^{-1}f_V^{-1}$, $f_*i_{U*} \simeq i_{V*}f_{V*}$ and $i_U^{-1}f^{-1} \simeq i_V^{-1}f_V^{-1}$. ☐

One can sheafify Theorem 8.7.3

Proposition 8.7.12. *Let $f\colon X \to Y$ be a morphism of sites. For $F \in \mathrm{Sh}(X,\mathscr{A})$ and $G \in \mathrm{Sh}(Y,\mathscr{A})$, there is a natural isomorphism in $\mathrm{Sh}(Y,\mathscr{A})$*

$$\mathscr{H}om\,(G, f_*F) \xrightarrow{\sim} f_*\mathscr{H}om\,(f^{-1}G, F). \tag{8.7.9}$$

Proof. Let $V \in \mathscr{C}_Y$ and set $U = f^t(V)$. Denote by $f_U\colon U \to V$ the morphism of sites associated with f. Using Proposition 8.7.11, we get the chain of isomorphisms

$$\begin{aligned}\Gamma(V; f_*\mathscr{H}om\,(f^{-1}G, F)) &\simeq \Gamma(U; \mathscr{H}om\,(f^{-1}G, F))\\ &\simeq \mathrm{Hom}\,(f^{-1}G|_U, F|_U)\\ &\simeq \mathrm{Hom}\,((f_U)^{-1}(G|_V), F|_U)\\ &\simeq \mathrm{Hom}\,(G|_V, (f_U)_*F|_U)\\ &\simeq \mathrm{Hom}\,(G|_V, (f_*F)|_V)\\ &\simeq \Gamma(V; \mathscr{H}om\,(G, f_*F)).\end{aligned}$$

These isomorphisms being functorial with respect to V, the isomorphism (8.7.9) follows. ☐

In the next statement, we assume $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$.

Proposition 8.7.13. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_U)$. One has the isomorphisms*

$$i_{U!}(G \otimes i_U^{-1}F) \simeq i_{U!}G \otimes F. \tag{8.7.10}$$

Proof. The right hand side of (8.7.10) is the sheaf associated with the presheaf

$$V \mapsto (\bigoplus_{s\colon V \to U} G(V)) \otimes F(V),$$

and the left hand side is the sheaf associated with the presheaf

$$V \mapsto \bigoplus_{s\colon V \to U} (G(V)) \otimes F(V)).$$

☐

Internal hom

We have already proved in Theorem 8.5.4 that for $F \in \mathrm{PSh}(X,\mathscr{A})$ and $G \in \mathrm{Sh}(X,\mathscr{A})$, the presheaf $\mathscr{H}om\,(F,G)$ is a sheaf. Moreover, it follows from (8.3.9) that the functor

$$\mathscr{H}om\,:\,(\mathrm{Sh}(X,\mathscr{A}))^{\mathrm{op}} \times \mathrm{Sh}(X,\mathscr{A}) \to \mathbf{Set}$$

is left exact. When $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$, the same result holds, this bi-functor taking its values in $\mathrm{Mod}(\mathbf{k}_X)$.

Finally, notice that (8.6.2) gives

$$\mathscr{H}om\,(F^a, G) \xrightarrow{\sim} \mathscr{H}om\,(F, G). \tag{8.7.11}$$

Tensor products

Here, we assume that $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. Recall Definition 8.3.11

Definition 8.7.14. Let $F_1, F_2 \in \mathrm{Sh}(\mathbf{k}_X)$. Their tensor product, denoted $F_1 \otimes F_2$ is the sheaf associated with the presheaf $F_1 \overset{\mathrm{psh}}{\otimes} F_2$.

Clearly, the bifunctor

$$\otimes\colon (\mathrm{Mod}(\mathbf{k}_X))^{\mathrm{op}} \times \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$$

is right exact. If $\mathbf{k}$ is a field, this functor is exact.

Proposition 8.7.15. *Let $F_i \in \mathrm{PSh}(k_X)$, $(i = 1, 2, 3)$. There is a natural isomorphism:*

$$\mathscr{H}om\,(F_1 \otimes F_2, F_3) \simeq \mathscr{H}om\,(F_1, \mathscr{H}om\,(F_2, F_3)).$$

Proof. This follows immediately from Proposition 8.3.12. □

8.8 Locally constant sheaves and glueing of sheaves

Locally constant sheaves

Definition 8.8.1. Let X be a site and let $M \in \mathscr{A}$.

(a) Let $M \in \mathscr{A}$. One denotes by M_X the sheaf associated with the constant presheaf $U \mapsto M$ and calls M_X the constant sheaf on X associated with M (or, when X is a topological space, "with stalk M").

(b) A constant sheaf is a sheaf isomorphic to a sheaf M_X for some $M \in \mathscr{A}$.

(c) A sheaf F on X is locally constant if there exists a covering $\mathscr{S} \in \mathrm{Cov}(X)$ such that $F|_U$ is a constant sheaf on U for each $U \in \mathscr{S}$.

(d) If $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$ and $\mathbf{k}$ is a field, a local system over $\mathbf{k}$ is a locally constant sheaf of finite rank (*i.e.*, locally isomorphic to $\mathbf{k}_X^m$ for some integer m).

If X is a topological space, the constant sheaf M_X is the sheaf of locally constant M-valued functions on X.

Locally constant sheaves, and their generalization, constructible sheaves, play an important role in various fields of mathematics.

Examples 8.8.2. (i) Let $X = \mathbb{R}$, the real line with coordinate t. The sheaf $\mathbb{C}_X \cdot \exp(t)$ of functions which are locally a constant multiple of the function $t \mapsto \exp(t)$ is isomorphic to the sheaf $\mathbb{C}_X$, hence is a constant sheaf.
(ii) Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z. Consider the differential operator $P = z\frac{\partial}{\partial z} - \alpha$, where $\alpha \in \mathbb{C} \setminus \mathbb{Z}$. Let us denote by K_α the kernel of P acting on $\mathscr{O}_X$.

Let U be an open disk in X centered at z_0, and let $A(z)$ denote a primitive of α/z in U. We have a commutative diagram of sheaves on U:

$$\begin{tikzcd}
\mathscr{O}_X \arrow[r, "z\frac{\partial}{\partial z}-\alpha"] \arrow[d, "\exp(-A(z))"'] & \mathscr{O}_X \arrow[d, "\frac{1}{z}\exp(-A(z))"] \\
\mathscr{O}_X \arrow[r, "\frac{\partial}{\partial z}"] & \mathscr{O}_X
\end{tikzcd}$$

Therefore, one gets an isomorphism of sheaves $K_\alpha|_U \xrightarrow{\sim} \mathbb{C}_X|_U$, which shows that K_α is locally constant, of rank one.

On the other hand, $f \in \mathscr{O}(X)$ and $Pf = 0$ implies $f = 0$. Hence $\Gamma(X; K_\alpha) = 0$, and K_α is a locally constant sheaf of rank one on $\mathbb{C} \setminus \{0\}$ which is not constant.
(iii) With the notations of Example 8.7.7 (iii), the sheaf $f_*\mathbf{k}_X$ is locally constant of rank 2.

We shall construct locally constant sheaves by glueing constant sheaves.

Glueing sheaves

One often encounters sheaves which are only defined locally, and it is natural to try to glue them.

For notational convenience, we shall often denote by $\mathscr{S} = \{U_i\}_{i\in I}$ a covering of $U \in \mathscr{C}_X$ indexed by a small set I (see Remark 8.4.2). (Recall that an object $V \in \mathscr{S}$ is a morphism $V \to U$ in $\mathscr{C}_X$.) In this case, we set

$$U_{ij} = U_i \times_U U_j, \quad U_{ijk} = U_i \times_U U_j \times_U U_k, \text{ etc.} \tag{8.8.1}$$

Consider a site X. Assume to be given a sheaf F and for each $i \in I$, a sheaf F_i on U_i and an isomorphism $\theta_i \colon F|_{U_i} \xrightarrow{\sim} F_i$. Set $\theta_{ji} := \theta_j \circ \theta_i^{-1}$. Then the family of isomorphisms of $\mathbf{k}_{U_{ij}}$-modules

$$\theta_{ji} \colon F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$$

will satisfy the condition

$$\theta_{ij} \circ \theta_{jk} = \theta_{ik} \text{ on } U_{ijk} \text{ for all } U_i, U_j, U_k \in \mathscr{S}. \tag{8.8.2}$$

We shall show that conversely, a family of isomorphisms θ_{ij} satisfying (8.8.2) permits us to reconstruct the sheaf F.

Theorem 8.8.3. *Let $\mathscr{S} = \{U_i\}_i$ be a covering of X. Assume to be given an object $F_i \in \mathrm{Sh}(U_i, \mathscr{A})$ for each $U_i \in \mathscr{S}$ and an isomorphism $\theta_{ji} \colon F_i|_{U_{ij}} \xrightarrow{\sim} F_j|_{U_{ij}}$ for each pair $U_i, U_j \in \mathscr{S}$, these isomorphisms satisfying the condition (8.8.2). Then there exists a sheaf $F \in \mathrm{Sh}(X, \mathscr{A})]$ and isomorphisms $\theta_i \colon F|_{U_i} \xrightarrow{\sim} F_i$ for $U_i \in \mathscr{S}$ such that $\theta_j = \theta_{ji} \circ \theta_i$ for $U_i, U_j \in \mathscr{S}$. Moreover, $(F, \{\theta_i\}_i)$ is unique up to unique isomorphism.*

The family of isomorphisms $\{\theta_{ij}\}$ satisfying conditions (8.8.2) is called a 1-cocycle.

Proof. (i) Unicity. Let $\theta_i\colon F|_{U_i} \xrightarrow{\sim} F_i$ and $\lambda_i\colon G|_{U_i} \xrightarrow{\sim} F_i$. Hence, $\theta_j = \theta_{ji} \circ \theta_i$ and $\lambda_j = \theta_{ji} \circ \lambda_i$ on U_j. Consider the isomorphisms

$$\rho_i := \lambda_i^{-1} \circ \theta_i\colon F|_{U_i} \to G|_{U_i}.$$

On U_{ij} we have:

$$\begin{aligned}\rho_j &= \lambda_j^{-1} \circ \theta_j = \lambda_j^{-1} \circ \theta_{ji} \circ \theta_i \\ &= \lambda_i^{-1} \circ \theta_i = \rho_i.\end{aligned}$$

Therefore, the isomorphisms ρ_i's will glue as a unique isomorphism $\rho\colon G \xrightarrow{\sim} F$ on X, by Theorem 8.5.4 and Corollary 8.5.5.

(ii) Existence of a presheaf F. For each open subset V of X, define $F(V)$ by the exact sequence

$$\begin{tikzcd} F(V) \arrow[r] & \prod_{i\in I} F_i(U_i \times_X V) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod_{j,k\in I} F_j(U_{jk} \times_X V). \end{tikzcd}$$

Here, the two arrows a, b are defined as follows. Let $U_j, U_k \in \mathscr{S}$. Then a is associated with the composition

$$\prod_{i\in I} F_i(U_i \times_X V) \to F_j(U_j \times_X V) \to F_j(U_{jk} \times_X V)$$

and b is associated with the composition

$$\prod_{i\in I} F_i(U_i \times_X V) \to F_k(U_k \times_X V) \to F_k(U_{jk} \times_X V) \xrightarrow{\theta_{jk}} F_j(U_{jk} \times_X V).$$

(iii) F is a sheaf. Indeed, let $V \in \mathscr{C}_X$ and let $\mathscr{V} \in \mathrm{Cov}(V)$. We may assume that $\mathscr{V}$ is stable by fiber products (see Remark 8.5.2). Then

$$F_i(U_i \times_X V) \xrightarrow{\sim} \lim_{W\in\mathscr{T}} F_i(U_i \times_X W),$$

and similarly with $F_j(U_{jk} \times_X V)$. Since products commute with projective limits, we get the isomorphism $F(V) \xrightarrow{\sim} \lim\limits_{W\in\mathscr{V}} F(W) = F(\mathscr{V})$.

(iv) The morphisms θ_i's are induced by the projections $F(V) \to \prod_{j\in I} F_j(V \times_X U_j) \to F_i(V \times_X U_i)$. Let us prove they are isomorphisms. Let $l \in I$. We can construct a commutative diagram

$$\begin{tikzcd} F_l(U_l) \arrow[r, "\alpha"] & \prod_{i\in I} F_i(U_i \times_X U_l) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, equal] & \prod_{j,k\in I} F_j(U_{jk} \times_X U_l) \arrow[d, equal] \\ F(U_l) \arrow[u, "\theta_l(U_l)"'] \arrow[r] & \prod_{i\in I} F_i(U_i \times_X U_l) \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & \prod_{j,k\in I} F_j(U_{jk} \times_X U_l) \end{tikzcd}$$

where $\alpha = \{\theta_{il}\}_i$. Since F_l is a sheaf on V_l, the sequence on the top is exact. The sequence on the bottom is exact by construction of F. It follows that $\theta_l(U_l)\colon F(U_l) \to F_l(U_l)$ is an isomorphism. Replacing U_l with any $V \to U_l$, we get the result. □

Example 8.8.4. Here, $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. Denote by $\mathbf{k}^\times$ the multiplicative group of invertible elements of $\mathbf{k}$. Let $X = \mathbb{S}^1$ be the 1-sphere (the circle), and consider a covering of X by two open connected intervals U_1 and U_2. Let $U_{12}^{\pm}$ denote the two connected components of $U_1 \cap U_2$. Let $\alpha \in \mathbf{k}^\times$. One defines a locally constant sheaf L_α on X of rank one over $\mathbf{k}$ by glueing $\mathbf{k}_{U_1}$ and $\mathbf{k}_{U_2}$ as follows. Let $\theta_\varepsilon : \mathbf{k}_{U_1}|_{U_{12}^\varepsilon} \to \mathbf{k}_{U_2}|_{U_{12}^\varepsilon}$ $(\varepsilon = \pm)$ be defined by $\theta_+ = 1$, $\theta_- = \alpha$. It follows from Theorem 8.8.3 that the locally constant sheaf L_α is well-defined. If $\alpha \neq 1$, this sheaf has no global section other than 0 on X.

If $\mathbf{k} = \mathbb{C}$ there is a more intuitive description of the sheaf L_α. Let us identify $\mathbb{S}^1$ with $[0, 2\pi]/\sim$, where $\sim$ is the relation which identifies 0 and 2π and let t denotes the coordinate. Choose $\beta \in \mathbb{C}$ such that $\exp(i\beta) = \alpha$. Then $L_\alpha \simeq \mathbb{C}_X \cdot exp(i\beta t)$.

Example 8.8.5. Here, $\mathscr{A} = \mathrm{Mod}(\mathbb{Z})$. Consider an n-dimensional real manifold X of class $\mathscr{C}^\infty$, and let $\{X_i, f_i\}_{i\in I}$ be an atlas. Recall what it means. The family $\{X_i\}_{i\in I}$ is an open covering of X and $f_i : X_i \xrightarrow{\sim} U_i$ is a topological isomorphism with an open subset U_i of $\mathbb{R}^n$ such that, setting $U_{ij}^i = f_i(X_{ij}) \subset \mathbb{R}^n$, the maps

$$f_{ji} := f_j|_{X_{ij}} \circ f_i^{-1}|_{U_{ij}^i} : U_{ij}^i \to U_{ij}^j, \tag{8.8.3}$$

are isomorphisms of class $\mathscr{C}^\infty$.

$$\begin{tikzcd}
 & X & X_i \arrow[l] \arrow[dl, "\sim"', "f_i"] & X_{ij} \arrow[l] \arrow[r] \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & X_j \arrow[r] \arrow[dr, "\sim", "f_j"'] & X & \\
\mathbb{R}^n & U_i \arrow[l] & U_{ij}^i \arrow[l] \arrow[rr, "\sim", "f_{ji}"'] & & U_{ij}^j \arrow[r] & U_j \arrow[r] & \mathbb{R}^n
\end{tikzcd}$$

The maps f_{ji} are called the transition functions. The locally constant function on X_{ij} defined as the sign of the Jacobian determinant of the f_{ji}'s is a 1-cocycle. It defines a sheaf locally isomorphic to $\mathbb{Z}_X$ called the orientation sheaf on X and denoted by or_X.

Exercises to Chapter 8

Exercise 8.1. Let $M \in \mathscr{A}$ and let X be a site. Recall that M_X denotes the sheaf associated with the constant presheaf $U \mapsto M$. Assume that X has a terminal object again denoted by X and denote by $b_X \colon X \to \mathrm{Pt}$ the canonical morphism of sites (see Example 8.4.6). Prove that

$$M_X \simeq b_X^{-1} M_{\mathrm{Pt}}.$$

(Recall that if X is a topological space and pt is the set with one element endowed with its natural topology, one also have $M_X \simeq a_X^{-1} M_{\mathrm{pt}}$.)

Exercise 8.2. Let X be a presite. Prove that a morphism $u \colon A \to B$ in $\mathrm{PSh}(X)$ is a monomorphism (resp. an epimorphism) if and only if the morphism $u(U) \colon A(U) \to B(U)$ in **Set** is a monomorphism (resp. an epimorphism) for all $U \in \mathscr{C}_X$.

Exercise 8.3. Let X be a presite. Consider morphisms $u \colon F \to H$ and $v \colon G \to H$ in $\mathrm{PSh}(X)$. Prove that for $U \in \mathscr{C}_X$, $(F \times_H G)(U) \simeq F(U) \times_{H(U)} G(U)$.

Exercise 8.4. Assume X is a topological space and let $U \in \mathrm{Op}_X$. Prove that the composition of morphisms of presites $U \xrightarrow{i_U} X \xrightarrow{j_U} U$ is isomorphic to the identity functor of the presite U. Show that this result is no more true in general.

Exercise 8.5. Let X be a presite satisfying (8.3.4) and let X_{fin} be the presite X endowed with the final topology. Prove the equivalence of categories $\mathrm{PSh}(X) \simeq \mathrm{Sh}(X_{\mathrm{fin}})$.

Exercise 8.6. Let $\alpha\colon \mathscr{J} \to \mathscr{I}$ be a functor of small categories and let $\mathscr{A}$ be a category which admits small colimits. Define the functor $\alpha_*\colon \mathrm{Fct}(\mathscr{I},\mathscr{A}) \to \mathrm{Fct}(\mathscr{J},\mathscr{A})$ by setting $\alpha_*(F) = F \circ \alpha$, $F \in \mathrm{Fct}(\mathscr{I},\mathscr{A})$.
(i) Prove that α_* admits a left adjoint.
(ii) Let $F\colon \mathscr{C} \to \mathscr{A}$ be a functor. We assume that $\mathscr{C}$ is small and recall that $\mathscr{A}$ admits small colimits. Prove that there exists a unique (up to isomorphism) functor $\widehat{F}\colon \mathscr{C}^\wedge \to \mathscr{A}$ which extends F and which commutes with small colimits in $\mathscr{C}^\wedge$.

Exercise 8.7. Prove isomorphism (8.7.5).

Exercise 8.8. Let X be a topological space. One defines the site X^f as follows. One sets $\mathscr{C}_{X^f} = \mathrm{Op}_X$ and one says that a small family of open sets $\{U_i\}_{i\in I}$ of X is an covering of an open set U in X^f if $U_i \subset U$ for all i and there exists a finite subset J of I such that $\bigcup_{j\in J} U_j = U$. One calls a covering in X^f an f-covering.

(i) Prove that the family of f-coverings defines a Grothendieck topology on X^f and that there is a natural morphism of sites $f\colon X \to X^f$.

(ii) Prove that the presheaf $\mathscr{C}_X^{0,b}$ of $\mathbb{C}$-valued continuous bounded functions is a sheaf on the site X^f and calculate $f^{-1}\mathscr{C}_X^{0,b}$.

Exercise 8.9. Let $\mathbf{k}$ be a field and X a connected topological space. Let L be a locally constant sheaf of rank one on X and set $L^{\otimes -1} := \mathscr{H}om\,(L, \mathbf{k}_X)$.

(i) Prove the isomorphism $L \otimes L^{\otimes -1} \simeq \mathbf{k}_X$.

(ii) Assume that there exists $s \in \Gamma(X; L)$ with $s \neq 0$. Prove that s defines an isomorphism $\mathbf{k}_X \xrightarrow{\sim} L$.

Chapter 9

Derived categories of abelian sheaves

We apply the results of Chapter 8 to the case of abelian sheaves, that is, when $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$. We prove in particular that $\mathrm{Mod}(\mathbf{k}_X) = \mathrm{Sh}(X, \mathrm{Mod}(\mathbf{k}))$ is a Grothendieck category.

Then we study injective and flat sheaves in order to introduce the derived category $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ of $\mathrm{Mod}(\mathbf{k}_X)$ and to construct the derived operations on sheaves.

We also introduce ringed sites and rapidly show that the category of modules over a sheaf of rings has essentially the same properties than that of abelian sheaves. We end this chapter with a brief introduction to ringed sites.

Notation. Recall that $\mathbf{k}$ denotes a commutative unital ring with finite global dimension (see [Wei94, 4.1.2]). As already mentioned, we shall write $\otimes$ instead of $\otimes_{\mathbf{k}}$, Hom instead of $\mathrm{Hom}_{\mathbf{k}}$. We proceed similarly with $\mathbf{k}$ replaced with $\mathbf{k}_X$ and for the derived functors.

Some references: [SGA4, KS06].

9.1 Abelian sheaves

In Sections 9.1–9.3, we assume that $\mathscr{A} = \mathrm{Mod}(\mathbf{k})$ and that X is a site satisfying (8.3.4). Recall the notations $\mathrm{Mod}(\mathbf{k}_X) = \mathrm{Sh}(X, \mathrm{Mod}(\mathbf{k}))$ and $\mathrm{Hom}_{\mathbf{k}_X} = \mathrm{Hom}_{\mathrm{Mod}(\mathbf{k}_X)}$.

The sheaf $\mathbf{k}_X$

Recall that $\mathbf{k}_X$ denotes the sheaf associated with the constant presheaf $\widetilde{\mathbf{k}}_X \colon U \mapsto \mathbf{k}$ and that this sheaf is called the constant sheaf on X with stalk $\mathbf{k}$.

For $U \in \mathscr{C}_X$, one sets for short $\mathbf{k}_{XU} := (\mathbf{k}_X)_U$.

Proposition 9.1.1. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and let $U \in \mathscr{C}_X$. One has the isomorphisms*

$$F_U \simeq \mathbf{k}_{XU} \otimes F, \tag{9.1.1}$$

$$\mathscr{H}om\,(\mathbf{k}_X, F) \simeq F, \quad \mathscr{H}om\,(\mathbf{k}_{XU}, F) \simeq \Gamma_U F. \tag{9.1.2}$$

Proof. (i) The isomorphism (9.1.1) is a particular case of Proposition 8.7.13.

(ii) The first isomorphism in (9.1.2) follows from Definition 8.3.10 and isomorphism 8.7.11. Let us prove the second isomorphism.

For $G \in \mathrm{Mod}(\mathbf{k}_X)$, one has the chain of isomorphisms (using Proposition 8.7.15):

$$\begin{aligned}\mathrm{Hom}_{\mathbf{k}_X}(G, \mathscr{H}om\,(\mathbf{k}_{XU}, F)) &\simeq \mathrm{Hom}_{\mathbf{k}_X}(G \otimes \mathbf{k}_{XU}, , F)\\ &\simeq \mathrm{Hom}_{\mathbf{k}_X}(G_U, F) \simeq \mathrm{Hom}_{\mathbf{k}_X}(j_U^{-1} i_U^{-1} G, F)\\ &\simeq \mathrm{Hom}_{\mathbf{k}_X}(G, i_{U*} i_U^{-1} F) \simeq \mathrm{Hom}_{\mathbf{k}_X}(G, \Gamma_U F).\end{aligned}$$

Since these isomorphisms are functorial with respect to G, the result follows from the Yoneda Lemma. ☐

Applying the functor $\Gamma(X; \bullet\,)$ to the second isomorphism (9.1.2) and using (8.7.5), we get that for F and U as above, one has

$$F(U) \simeq \mathrm{Hom}_{\mathbf{k}_X}(\mathbf{k}_{XU}, F). \tag{9.1.3}$$

Recall Definition 5.4.1.

Proposition 9.1.2. *the family* $\{\mathbf{k}_{XU}\}_{U \in \mathscr{C}_X}$ *is a system of generators of the category* $\mathrm{Mod}(\mathbf{k}_X)$.

Proof. A morphism of sheaves $\varphi\colon F \to G$ is an isomorphism as soon as for each $U \in \mathscr{C}_X$, $\varphi(U)\colon F(U) \to G(U)$ is an isomorphism.

Then the result follows from (9.1.3). ☐

Remark 9.1.3. When X is a topological space, one better uses the functors i_U^{-1} and $i_{U!}$ rather than j_{U*} and j_U^{-1}, respectively.

If U, V are open subsets, then $U \times_X V = U \cap V$. It follows that the morphism of sites i_U corresponds to the continuous embedding $U \hookrightarrow X$. Since the composition of morphisms of sites

$$U \xrightarrow{i_U} X \xrightarrow{\mathrm{j}_U} U \tag{9.1.4}$$

is the identity, we obtain:

$$i_U^{-1} \circ i_{U*} \simeq \mathrm{id},\ i_U^{-1} \circ i_{U!} \simeq \mathrm{id}\,. \tag{9.1.5}$$

Hence, i_{U*} and $i_{U!}$ are fully faithful in this case.

Lemma 9.1.4. *Let* $\varphi\colon F \to G$ *be a morphism in* $\mathrm{Mod}(\mathbf{k}_X)$. *Denote by* “$\mathrm{Im}$”$\varphi$ *and* “Coim”φ *the image and coimage of this morphism in the category* $\mathrm{PSh}(\mathbf{k}_X)$ *(i.e., the image and coimage of* $\iota_X(\varphi)$*). Then* $\mathrm{Im}\,\varphi \simeq (\text{“}\mathrm{Im}\text{”}\varphi)^a$ *and* $\mathrm{Coim}\,\varphi \simeq (\text{“}\mathrm{Coim}\text{”}\varphi)^a$.

Proof. By Theorem 8.6.6, the category $\mathrm{Mod}(\mathbf{k}_X)$ admits small limits and colimits. Denote by “$\oplus$” and “Coker” the coproduct and the cokernel in the category $\mathrm{PSh}(\mathbf{k}_X)$. Then (recall Lemma 5.1.1):

$$\begin{aligned}\mathrm{Coim}\,\varphi &= \mathrm{Coker}\big(F \times_G F \rightrightarrows F\big)\\ &\simeq \big(\text{“}\mathrm{Coker}\text{”}\big(F \times_G F \rightrightarrows F\big)\big)^a \simeq \big(\text{“}\mathrm{Coim}\text{”}(\varphi)\big)^a,\\ \mathrm{Im}\,\varphi &= \ker\big(G \rightrightarrows G \oplus_F G\big) \simeq \ker\big(G \rightrightarrows (G \underset{F}{\text{“}\oplus\text{”}} G)^a)\big)\\ &\simeq \big(\ker(G \rightrightarrows G \text{“}\oplus\text{”} G)\big)^a \simeq \big(\text{“}\mathrm{Im}\text{”}(\varphi)\big)^a.\end{aligned}$$

Here, the fourth isomorphism follows from the fact that the functor ^{a} being exact, it commutes with kernels. It follows that for a morphism $\varphi\colon F \to G$ in $\mathrm{Mod}(\mathbf{k}_X)$, the natural morphism $\mathrm{Coim}\,\varphi \to \mathrm{Im}\,\varphi$ is an isomorphism. ☐

Theorem 9.1.5. *The category* $\mathrm{Mod}(\mathbf{k}_X)$ *is an abelian Grothendieck category.*

Proof. The additive category $\mathrm{Mod}(\mathbf{k}_X)$ is abelian thanks to Theorem 8.6.6 and Lemma 9.1.4. It admits a small system of generators thanks to Proposition 9.1.2 ☐

Remark 9.1.6. The object $\mathscr{G} := \bigoplus_{U\in\mathscr{C}_X} \mathbf{k}_{XU}$ is a generator of the abelian category $\mathrm{Mod}(\mathbf{k}_X)$.

Corollary 9.1.7. *Let* $\varphi\colon F\to G$ *be a morphism in* $\mathrm{Mod}(\mathbf{k}_X)$. *Then* φ *is an epimorphism if and only if, for any* $U\in\mathscr{C}_X$ *and any* $t\in G(U)$, *there exists* $\mathscr{S}\in\mathrm{Cov}(U)$ *and for any* $V\in\mathscr{S}$ *there exists* $s\in F(V)$ *with* $\varphi(s)=t$ *in* $G(V)$.

Proof. As above, denote by "Im"φ the image of φ in the category $\mathrm{PSh}(\mathbf{k}_X)$. Since this presheaf is a subpresheaf of the sheaf G, it is separated. Hence $\mathrm{Im}(\varphi)\simeq(\text{“Im”}(\varphi))^+$ and

$$\mathrm{Im}(\varphi)(U)\simeq \operatorname*{colim}_{\mathscr{S}\in\mathrm{Cov}(U)} \text{“Im”}(\varphi)(\mathscr{S}).$$

Now φ is an epimorphism if and only if $\mathrm{Im}\,\varphi\xrightarrow{\sim}G$. Let $t\in G(U)$. Since $\mathrm{Cov}(U)^{\mathrm{op}}$ is filtered, $t\in\mathrm{Im}(\varphi)(U)$ if and only if there exists $\mathscr{S}\in\mathrm{Cov}(U)$ with $t\in\text{“Im”}(\varphi)(\mathscr{S})$. The result follows. ☐

Corollary 9.1.8. *Let* $F'\xrightarrow{\varphi}F\xrightarrow{\psi}F''$ *be a complex in* $\mathrm{Mod}(\mathbf{k}_X)$. *Then the conditions below are equivalent:*

(a) *this complex is exact,*

(b) *for any* $U\in\mathscr{C}_X$ *and any* $s\in F(U)$ *such that* $\psi(s)=0$, *there exist a covering* $\mathscr{S}\in\mathrm{Cov}(U)$ *and for each* $V\in\mathscr{S}$ *there exists* $t\in F'(V)$ *such that* $\varphi(t)=s|_V$,

(c) *there exists a covering* $\mathscr{S}$ *of* X *such that the sequence* $F'|_U\xrightarrow{\varphi}F|_U\xrightarrow{\psi}F''|_U$ *is exact for any* $U\in\mathscr{S}$,

(d) *for any covering* $\mathscr{S}$ *of* X *and any* $U\in\mathscr{S}$, *the sequence* $F'|_U\xrightarrow{\varphi}F|_U\xrightarrow{\psi}F''|_U$ *is exact.*

Proof. (a) is equivalent to saying that the natural morphism $\mathrm{Im}\,\varphi\to\ker\psi$ is an epimorphism, and this last condition is equivalent to (b) by Corollary 9.1.7.

(a) ⇔ (c) ⇔ (b) follow from Corollary 8.5.5 and the isomorphisms, for $U\in\mathscr{C}_X$, $(\mathrm{Im}\,\varphi)|_U\simeq\mathrm{Im}(\varphi|_U)$, $(\ker\psi)|_U\simeq\ker(\psi|_U)$. ☐

Corollary 9.1.9. *Assume that* X *is a topological space. Then a complex* $F'\to F\to F''$ *in* $\mathrm{Mod}(\mathbf{k}_X)$ *is exact if and only if the sequence* $F'_x\to F_x\to F''_x$ *is exact in* $\mathrm{Mod}(\mathbf{k})$ *for any* $x\in X$.

Examples 9.1.10. Recall Examples 8.1.7.

(i) Let X be a real a manifold of class C^∞ and dimension n. The augmented de Rham complex is the complex

$$0\to\mathbb{C}_X\to\mathscr{C}_X^{\infty,(0)}\xrightarrow{d}\cdots\to\mathscr{C}_X^{\infty,(n)}\to0 \tag{9.1.6}$$

where d is the differential. This complex of sheaves is exact. In other words, the sheaf $\mathbb{C}_X$ is quasi-isomorphic to the de Rham complex

$$0 \to \mathscr{C}_X^{\infty,(0)} \xrightarrow{d} \cdots \to \mathscr{C}_X^{\infty,(n)} \to 0. \tag{9.1.7}$$

The same result holds with the sheaf $\mathscr{C}_X^{\infty}$ replaced with the sheaf $\mathscr{D}b_X$ or, assuming that X is real analytic, with the sheaf $\mathscr{C}_X^{\omega}$ or the sheaf $\mathscr{B}_X$.

(ii) Let X be a complex manifold of dimension n. The augmented holomorphic de Rham complex is

$$0 \to \mathbb{C}_X \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0 \tag{9.1.8}$$

where d is the holomorphic differential and, as already mentioned, Ω_X^p denotes the sheaf of holomorphic p-forms. In particular, $\Omega_X^0 = \mathscr{O}_X$. This complex of sheaves is exact. In other words, the sheaf $\mathbb{C}_X$ is quasi-isomorphic to the holomorphic de Rham complex

$$0 \to \Omega_X^0 \xrightarrow{d} \cdots \to \Omega_X^n \to 0. \tag{9.1.9}$$

9.2 Flat sheaves and injective sheaves

Let X be a site. A sheaf on X is injective (resp. projective) if it is an injective (resp. projective) object of $\mathrm{Mod}(\mathbf{k}_X)$.

Flat sheaves

Definition 9.2.1. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. One says that F is flat if the functor $F \otimes \bullet$ is exact.

Although, in general, the category $\mathrm{Mod}(\mathbf{k}_X)$ does not have enough projectives (see [KS90, Exe. II. 23]), it admits enough flat objects and Proposition 9.2.4will allow us to derive the tensor product.

Lemma 9.2.2. *Let $U \in \mathscr{C}_X$. Then the sheaf $\mathbf{k}_{XU}$ is flat.*

Proof. We have already proved that for $F \in \mathrm{Mod}(\mathbf{k}_X)$, one has an isomorphism $F_U \simeq \mathbf{k}_{XU} \otimes F$ and that the functor $F \mapsto F_U$ is exact. □

Proposition 9.2.3. *Let $\{G_i\}_{i\in I}$ be either a small family or a filtered small inductive system of flat sheaves. Then $\bigoplus_{i\in I} G_i$ or $\operatorname{colim}_i G_i$ is flat.*

Proof. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, the functor $\bullet \otimes F$ commutes with small direct sums as well as with small filtered colimits. □

Recall Definition 7.2.1.

Proposition 9.2.4. *The category of flat sheaves is generating in* $\mathrm{Mod}(\mathbf{k}_X)$.

Proof. We have already seen in Remark 9.1.6 that $\mathscr{G} = \bigoplus_{U\in\mathscr{C}_X} \mathbf{k}_{XU}$ is a generator of $\mathrm{Mod}(\mathbf{k}_X)$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. By Lemma 5.4.2, there exists a small set I and an epimorphism $\mathscr{G}^{\oplus I} \twoheadrightarrow F$. Since the sheaves $\mathbf{k}_{XU}$ are flat and small direct sums of flat sheaves are flat, the sheaf $\mathscr{G}^{\oplus I}$ is flat. □

Recall Proposition 7.4.4 and Definition 7.4.2

Corollary 9.2.5. *Denote by $\mathscr{F}$ the full additive subcategory of* $\operatorname{Mod}(\mathbf{k}_X)$ *consisting of flat sheaves. Then for $G \in \operatorname{Mod}(\mathbf{k}_X)$, the category $\mathscr{F}$ is $(G \otimes \bullet)$-projective and $\mathscr{F} \times \mathscr{F}$ if $\otimes$-projective.*

Recall Definition **??** (and it extension to bifunctors).

Proposition 9.2.6. *Assume that X is a topological space. Then the right exact bifunctor $\otimes$ has finite cohomological dimension.*

Proof. For $x \in X$, $(F_1 \otimes F_2)_x \simeq (F_1)_x \otimes (F_2)_x$ and it follows that F is flat if and only if F_x is a $\mathbf{k}$-flat module. Moreover, since $(H^j(F_1 \otimes F_2))_x \simeq H^j((F_1 \otimes F_2)_x)$, we get that the cohomological dimension of the functor $F \otimes_{\mathbf{k}_X} \bullet$ $(F \in \operatorname{Mod}(\mathbf{k}_X))$ is the same as that of the functor $M \otimes_{\mathbf{k}} \bullet$ $(M \in \operatorname{Mod}(\mathbf{k}))$. This dimension is finite thanks to the hypothesis that $\mathbf{k}$ has finite global dimension. □

Proposition 9.2.7. *Let $f \colon X \to Y$ be a morphism of sites. Let $G \in \operatorname{Mod}(\mathbf{k}_Y)$ and assume that G is flat. Then $f^{-1}G$ is flat in* $\operatorname{Mod}(\mathbf{k}_X)$*.*

Proof. We shall only prove this result on topological spaces, referring to [KS06, Lem. 18.6.7] for the general case (which is more difficult). By Example 9.1.9, a complex $F' \to F \to F''$ in $\operatorname{Mod}(\mathbf{k}_X)$ is exact if and only if, for all $x \in X$, the sequence $F'_x \to F_x \to F''_x$ in $\operatorname{Mod}(\mathbf{k})$ is exact. Therefore, a sheaf H on X is flat if and only if H_x is a flat $\mathbf{k}$-module for all $x \in X$. Since $(f^{-1}G)_x \simeq G_{f(x)}$, the result follows. □

Injective sheaves

Since the category $\operatorname{Mod}(\mathbf{k}_X)$ is a Grothendieck category (Theorem 9.1.5), it admits enough injective objects by Theorem 5.4.4.

Proposition 9.2.8. *Let $F, G \in \operatorname{Mod}(\mathbf{k}_X)$.*

(a) *Assume that F is injective and G is flat. Then $\mathscr{H}om\,(G, F)$ is injective.*

(b) *Let $\{F_i\}_{i\in I}$ be a small family of injective sheaves. Then $\prod_{i\in I} F_i$ is injective.*

Proof. (a) follows from the isomorphism $\operatorname{Hom}(\bullet, \mathscr{H}om\,(G, F)) \simeq \operatorname{Hom}(\bullet \otimes G, F)$ since the functor on the right-hand side is exact.
(b) follows from the isomorphism $\operatorname{Hom}(\bullet, \prod_{i\in I} F_i) \simeq \prod_{i\in I} \operatorname{Hom}(\bullet, F_i)$ and the fact that $\prod_{i\in I}$ is exact on $\operatorname{Mod}(\mathbf{k})$. □

Proposition 9.2.9. *Let $f \colon X \to Y$ be a morphism of sites. Let $F \in \operatorname{Mod}(\mathbf{k}_X)$ and assume that F is injective. Then f_*F is injective in* $\operatorname{Mod}(\mathbf{k}_Y)$*. In particular, for $U \in \mathscr{C}_X$, $i_U^{-1}F$ is injective in* $\operatorname{Mod}(\mathbf{k}_U)$ *and $\Gamma_U F$ is injective in* $\operatorname{Mod}(\mathbf{k}_X)$*.*

Proof. (i) By hypothesis, the functor $\operatorname{Hom}_{\mathbf{k}_X}(\bullet, F)$ is exact on $\operatorname{Mod}(\mathbf{k}_X)$. Since the functor f^{-1} is exact , we get that the functor $\operatorname{Hom}_{\mathbf{k}_X}(f^{-1}(\bullet), F)$ is exact on $\operatorname{Mod}(\mathbf{k}_Y)$. Then the result follows by adjunction.
(ii) Since $i_U^{-1}F \simeq j_{U*}F$ and $\Gamma_U F = i_{U*} i_U^{-1} F$, the proof is complete. □

Proposition 9.2.10. *Let $F \in \operatorname{Mod}(\mathbf{k}_X)$ and assume that F is injective. Then the functor $\mathscr{H}om\,(\bullet, F)$ is exact.*

Proof. Let $U \in \mathscr{C}_X$ and let $G' \to G \to G''$ be an exact sequence in $\mathrm{Mod}(\mathbf{k}_X)$. Since $\Gamma_U F$ is injective, the sequence $\mathrm{Hom}\,(G'', \Gamma_U F) \to \mathrm{Hom}\,(G, \Gamma_U F) \to \mathrm{Hom}\,(G', \Gamma_U F)$ is exact. Then the result follows from the isomorphisms

$$\mathrm{Hom}\,(H, \Gamma_U F) \simeq \mathrm{Hom}\,(H_U, F) \simeq \mathrm{Hom}\,(\mathbf{k}_U, \mathscr{H}om\,(H, F)) \simeq \Gamma(U; \mathscr{H}om\,(H, F)).$$

□

Remark 9.2.11. Since $\mathrm{Mod}(\mathbf{k}_X)$ is a Grothendieck category, there are enough injective sheaves. However, such sheaves are not easy to construct. When X is a topological space and $\mathbf{k}$ is a field, one can show that flabby sheaves (see § 10.5) are injectives (see Exercise 10.13 below).

Example 9.2.12. Let X denote a real analytic manifold. The sheaf $\mathscr{B}_X$ of Sato's hyperfunctions is injective (as a $\mathbb{C}_X$-module), contrarily to the sheaf $\mathscr{D}b_X$ of Schwartz's distributions (see Exercise 10.9 below.)

9.3 The derived category of sheaves

Recall that $\mathrm{Mod}(\mathbf{k}_X)$ is an abelian Grothendieck category and, in particular, admits enough injectives (see Theorem 5.4.4). Hence we may derive all left exact functors defined on $\mathrm{Mod}(\mathbf{k}_X)$. Using Proposition 9.2.4 we get that $\mathrm{Mod}(\mathbf{k}_X)$ has enough flat objects and this allows us to derive the tensor product functor. We denote by $\mathrm{D}^*(\mathbf{k}_X)$ the derived category $\mathrm{D}^*(\mathrm{Mod}(\mathbf{k}_X))$, with $^* = +, -, \mathrm{b}, \mathrm{ub}$.

Notation 9.3.1. We have already encountered the left exact functors $\Gamma(U; \bullet)$ and $\Gamma(X; \bullet)$ (see Proposition 8.6.7). Classically, for a sheaf F one sets

$$H^j(X; F) = H^j\mathrm{R}\Gamma(X; F),$$

and similarly with U instead of X.

Applying the results of Part I, we get:

Theorem 9.3.2. *Let $f\colon X \to Y$ be a morphism of sites and let $U \in \mathscr{C}_X$. The functors below are well defined (here, we use the fact that the homological dimension of $\mathbf{k}$ is finite):*

$$\begin{aligned}
&\mathrm{RHom}\,(\bullet, \bullet)\colon \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}),\\
&\mathrm{R}\mathscr{H}om\,(\bullet, \bullet)\colon \mathrm{D}^-(\mathbf{k}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_X),\\
&f^{-1}\colon \mathrm{D}^*(\mathbf{k}_Y) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{R}f_*\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y),\\
&\mathrm{R}\Gamma(X; \bullet)\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}),\\
&\bullet \overset{\mathrm{L}}{\otimes} \bullet\colon \mathrm{D}^*(\mathbf{k}_X) \times \mathrm{D}^*(\mathbf{k}_X) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{j}_{U*} \simeq i_U^{-1}\colon \mathrm{D}^*(\mathbf{k}_X) \to \mathrm{D}^*(\mathbf{k}_U) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{j}_U^{-1} \simeq i_{U!}\colon \mathrm{D}^*(\mathbf{k}_U) \to \mathrm{D}^*(\mathbf{k}_X) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{R}i_{U*}\colon \mathrm{D}^+(\mathbf{k}_U) \to \mathrm{D}^+(\mathbf{k}_X).
\end{aligned}$$

There are other functors which are combinations or particular cases of the preceding ones, such as the functor $F \mapsto F_U$ defined on $\mathrm{D}^*(\mathbf{k}_X)$ ($* = \mathrm{b}, +, -$) or the right derived functor $\mathrm{R}\Gamma_U(\,\bullet\,)\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_X)$.

Then one can extend much of the preceding formulas to the derived functors and obtain the next important formulas.

Theorem 9.3.3. (a) *For $F_1, F_3 \in \mathrm{D}^-(\mathbf{k}_X)$ and $F_2 \in \mathrm{D}^+(\mathbf{k}_X)$ one has:*

$$\begin{aligned}&\text{(i)}\ \mathrm{RHom}\,(F_1, F_2) \simeq \mathrm{R}\Gamma(X; \mathrm{R}\mathscr{H}om\,(F_1, F_2)),\\ &\text{(ii)}\ \mathrm{R}\mathscr{H}om\,(F_1 \overset{\mathrm{L}}{\otimes} F_3, F_2) \simeq \mathrm{R}\mathscr{H}om\,(F_1, \mathrm{R}\mathscr{H}om\,(F_3, F_2)).\end{aligned}$$

(b) *Let $f\colon X \to Y$ be a morphism of sites and let $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G, G_1, G_2 \in \mathrm{D}^-(\mathbf{k}_Y)$. Then*

$$\begin{aligned}&\text{(i)}\ \mathrm{R}\mathscr{H}om\,(G, \mathrm{R}f_*F) \simeq \mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(f^{-1}G, F),\\ &\text{(ii)}\ f^{-1}G_1 \overset{\mathrm{L}}{\otimes} f^{-1}G_2 \simeq f^{-1}(G_1 \overset{\mathrm{L}}{\otimes} G_2).\end{aligned}$$

Proof. (a)–(i) One represents F_1 by a complex, bounded from above, of flat sheaves and F_2 by a complex, bounded from below, of injectives sheaves. Then the result is deduced from (9.1.3) and Proposition 9.2.8 (a).

(a)–(ii) The non derived formula is proved in Proposition 8.7.15. Then replace F_1 and F_3 with complexes bounded from above of flat sheaves and F_2 with a complex bounded from below of injectives sheaves. This gives the result thanks to Proposition 9.2.8.

(b)–(i) One replaces F with a bounded from below of injective sheaves and use the fact that direct images of injective sheaves are injective (Proposition 9.2.9 (a)).

(b)–(ii) Replace G_1 (or G_2) with a bounded from above complex of flat sheaves and use the fact that inverse images of flat sheaves are flat (Proposition 9.2.7). ☐

There are many other important formulas, such as:

$$\begin{aligned}&(i)\ \mathrm{Hom}_{\mathrm{D}^*(\mathbf{k}_X)}(\,\bullet\,,\,\bullet\,) \simeq H^0\mathrm{RHom}\,(\,\bullet\,,\,\bullet\,),\\ &(ii)\ (f \circ g)^{-1} \simeq g^{-1} \circ f^{-1}, \quad \mathrm{R}(f \circ g)_* \simeq \mathrm{R}f_* \circ \mathrm{R}g_*,\\ &(iii)\ \mathrm{R}\Gamma(U; F) \simeq \mathrm{RHom}\,(\mathbf{k}_{XU}, F).\end{aligned}$$

Indeed, (i) is a particular case of Theorem 7.4.9, the first isomorphism in (ii) is clear and the second follows by adjunction and, finally, (iii) follows from Proposition **??**.

Notation 9.3.4. In the literature, one often encounters the following notations:

$$\begin{aligned}&\mathscr{E}xt^j(F, G) = H^j(\mathrm{R}\mathscr{H}om\,(F, G)), \quad \mathrm{Ext}^j(F, G) = H^j(\mathrm{RHom}\,(F, G)),\\ &\mathscr{T}or_j(F, G) = H^{-j}(F \overset{\mathrm{L}}{\otimes} G), \quad \mathrm{Tor}_j(F, G) = H^{-j}(\mathrm{R}\Gamma(X; F \overset{\mathrm{L}}{\otimes} G)),\\ &H^j(X; F) = H^j(\mathrm{R}\Gamma(X; F))\ .\end{aligned}$$

9.4 Modules over sheaves of rings

It is possible to generalise the preceding constructions by replacing the constant sheaf $\mathbf{k}_X$ with a sheaf of rings $\mathscr{R}$. We shall only present here the main ideas of this theory, skipping some details.

A sheaf of $\mathbf{k}$-algebras (or, equivalently, a $\mathbf{k}_X$-algebra) $\mathscr{R}_X$ on a site X is a sheaf of $\mathbf{k}$-modules such that for each $U \in \mathscr{C}_X$, $\mathscr{R}_X(U)$ is endowed with a structure of a $\mathbf{k}$-algebra and the operations (addition, multiplication) commute to the restriction morphisms. If $\mathscr{R}_X$ is a sheaf of rings, $\mathscr{R}_X^{\mathrm{op}}$ is the sheaf of rings $U \mapsto \mathscr{R}_X(U)^{\mathrm{op}}$. A sheaf of $\mathbb{Z}$-algebras is simply called a sheaf of rings.

If $\mathscr{R}_X$ is a sheaf of rings, one defines in an obvious way the notion of a presheaf F of (left) $\mathscr{R}_X$-modules as follows: for each $U \in \mathscr{C}_X$, $F(U)$ is an $\mathscr{R}_X(U)$-module and the action of $\mathscr{R}_X(U)$ on $F(U)$ commutes to the restriction morphisms. One also naturally defines the notion of an $\mathscr{R}_X$-linear morphism of presheaves of $\mathscr{R}_X$-modules and we get the abelian category $\mathrm{PSh}(\mathscr{R}_X))$ of such presheaves.

A sheaf of $\mathscr{R}_X$-modules, or simply, an $\mathscr{R}_X$-module, is a presheaf of $\mathscr{R}_X$-modules which is a sheaf of $\mathbf{k}_X$-modules. Hence we have defined the category $\mathrm{Mod}(\mathscr{R}_X)$ of $\mathscr{R}_X$-modules. Note that the forgetful functor

$$for\colon \mathrm{Mod}(\mathscr{R}_X) \to \mathrm{Mod}(\mathbf{k}_X) \tag{9.4.1}$$

is faithful and conservative.

Moreover, the functor ${}^a\colon\ \mathrm{PSh}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$ induces a functor

$${}^a\colon\ \mathrm{PSh}(\mathscr{R}_X) \to \mathrm{Mod}(\mathscr{R}_X) \tag{9.4.2}$$

left adjoint to the embedding $\iota_X\colon \mathrm{Mod}(\mathscr{R}_X) \hookrightarrow \mathrm{PSh}(\mathscr{R}_X)$.

Examples 9.4.1. (i) Let R be a $\mathbf{k}$-algebra. The constant sheaf R_X is a sheaf of $\mathbf{k}$-algebras. In particular, $\mathbf{k}_X$ is a sheaf of $\mathbf{k}$-algebras.
(ii) On a topological space, the sheaf $\mathscr{C}_X^0$ is a $\mathbb{C}_X$-algebra. If X is a real differentiable manifold, the sheaf $\mathscr{C}_X^\infty$ is a $\mathbb{C}_X$-algebra. The sheaf $\mathscr{D}b_X$ is a $\mathscr{C}_X^\infty$-module.
(iii) If X is complex manifold, the sheaves $\mathscr{O}_X$ and $\mathscr{D}_X$ are $\mathbb{C}_X$-algebras and $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module.

The next result is obvious.

Lemma 9.4.2. *Let $u\colon F \to G$ be a morphism in $\mathrm{Mod}(\mathscr{R}_X)$ and denote by $for(u)$ its image by the functor for of (9.4.1). Then $\ker(for(u))$ and $\mathrm{Coker}(for(u))$ belong to $\mathrm{Mod}(\mathscr{R}_X)$ and are respectively a kernel and a cokernel of u in this category.*

More generally, the category $\mathrm{Mod}(\mathscr{R}_X)$ admits small limits and small colimits and these limits commute with the functor for.

By using this lemma, Theorem 9.1.5 extends naturally to the case of $\mathscr{R}_X$-modules.

Theorem 9.4.3. *Let $\mathscr{R}_X$ be a sheaf of $\mathbf{k}$-algebras on a site X.*

(i) *The category $\mathrm{Mod}(\mathscr{R}_X)$ admits small limits and such limits commute with the functor ι_X.*

(ii) *The category* $\mathrm{Mod}(\mathscr{R}_X)$ *admits small colimits. More precisely, if* $\{F_i\}_{i\in I}$ *is an inductive system of* $\mathscr{R}_X$*-modules, its colimit is the sheaf associated with its colimit in* $\mathrm{PSh}(\mathscr{R}_X)$*.*

(iii) *The functor* $\iota_X\colon \mathrm{Mod}(\mathscr{R}_X) \to \mathrm{PSh}(\mathscr{R}_X)$ *is fully faithful and commutes with small limits (in particular, it is left exact). The functor* ${}^a : \mathrm{PSh}(\mathscr{R}_X)) \to \mathrm{Mod}(\mathscr{R}_X)$ *commutes with small colimits and is exact.*

(iv) *Small filtered colimits are exact in* $\mathrm{Mod}(\mathscr{R}_X)$*.*

(v) *Let* $\varphi\colon F \to G$ *be a morphism in* $\mathrm{Mod}(\mathscr{R}_X)$*. Denote by* $\text{“}\mathrm{Im}\text{”}\varphi$ *and* $\text{“}\mathrm{Coim}\text{”}\varphi$ *the image and coimage of this morphism in the category* $\mathrm{PSh}(\mathscr{R}_X)$ (i.e., *the image and coimage of* $\iota_X(\varphi)$)*. Then* $\mathrm{Im}\,\varphi \simeq (\text{“}\mathrm{Im}\text{”}\varphi)^a$ *and* $\mathrm{Coim}\,\varphi \simeq (\text{“}\mathrm{Coim}\text{”}\varphi)^a$*.*

(vi) *The category* $\mathrm{Mod}(\mathscr{R}_X)$ *is an abelian Grothendieck category. Moreover, the functor* for *in* (9.4.1) *is faithful, exact and conservative. In particular, a complex* $F' \to F \to F''$ *in* $\mathrm{Mod}(\mathscr{R}_X)$ *is exact if and only if it is exact when applying the functor* for*.*

One defines naturally the functors

$$\begin{aligned}&\mathscr{H}\!om_{\mathscr{R}_X}\colon \mathrm{Mod}(\mathscr{R}_X)^{\mathrm{op}} \times \mathrm{Mod}(\mathscr{R}_X) \to \mathrm{Mod}(\mathbf{k}_X),\\ &\bullet \otimes_{\mathscr{R}_X} \bullet\colon \mathrm{Mod}(\mathscr{R}_X^{\mathrm{op}}) \times \mathrm{Mod}(\mathscr{R}_X) \to \mathrm{Mod}(\mathbf{k}_X).\end{aligned}$$

If $\mathscr{R}_X$ is a sheaf of commutative algebras, then these two functors take their values in $\mathrm{Mod}(\mathscr{R}_X)$.

Let $f\colon X \to Y$ be a morphism of sites and let $\mathscr{R}_Y$ be a $\mathbf{k}_Y$-algebra. Clearly, $f^{-1}\mathscr{R}_Y$ is a $\mathbf{k}_X$-algebra and the functors below are well defined and adjoint one to each other.

$$\begin{tikzcd}\mathrm{Mod}(f^{-1}\mathscr{R}_Y) \arrow[r, shift left, "f_*"] & \mathrm{Mod}(\mathscr{R}_Y). \arrow[l, shift left, "f^{-1}"]\end{tikzcd}$$

Moreover, the functor f^{-1} is exact, thanks to Theorem 9.4.3 (vi).

Definition 9.4.4. Let $\mathscr{R}_X$ be a sheaf of $\mathbf{k}$-algebras on the site X and let F be a sheaf of $\mathscr{R}_X$-modules on X.

(i) F is injective (resp. projective) if F is an injective (resp. a projective) object in the category $\mathrm{Mod}(\mathscr{R}_X)$.

(ii) F is flat if the functor $\bullet \otimes_{\mathscr{R}_X} F$ is exact,

Remark that, if X is a topological space, F is flat if and only if F_x is an $\mathscr{R}_{X,x}$-flat module for any $x \in X$.

Propositions 9.2.9 and 9.2.10 extend to $\mathscr{R}$-modules. In particular:

Proposition 9.4.5. *Let* $\mathscr{R}_Y$ *be a sheaf of* $\mathbf{k}$*-algebras on* Y *and let* $f\colon X \to Y$ *be a morphism of sites. Let* $F \in \mathrm{Mod}(f^{-1}\mathscr{R}_Y)$*. Then* f_*F *is injective in* $\mathrm{Mod}(\mathscr{R}_Y)$*.*

Proof. This follows immediately from the adjunction formula in Theorem 8.7.3 and the fact that the functor f^{-1} is exact. □

By the same proof as for Proposition 9.2.10, we get;

Proposition 9.4.6. *Let $F \in \mathrm{Mod}(\mathscr{R}_X)$ and assume that F is injective. Then the functor $\mathscr{H}om_{\mathscr{R}_X}(\,\bullet\,, F)$ is exact.*

Proposition 9.4.7. *Let $\mathscr{R}$ be a sheaf of $\mathbf{k}$-algebras on the site X and let $G \in \mathrm{Mod}(\mathscr{R}^{\mathrm{op}})$. Then the category of flat $\mathscr{R}$-modules is projective with respect to the functor $G \otimes_{\mathscr{R}_X} \bullet$.*

The proof goes as for Proposition 9.2.4.

Similarly as in Theorem 9.3.2, for $f\colon X \to Y$ a morphism of sites and for $U \in \mathscr{C}_X$, the functors below are well defined.

$$
\begin{aligned}
&\mathrm{RHom}_{\mathscr{R}_X}(\,\bullet\,,\,\bullet\,)\colon \mathrm{D}^-(\mathscr{R}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathscr{R}_X) \to \mathrm{D}^+(\mathbf{k}),\\
&\mathrm{R}\mathscr{H}om_{\mathscr{R}_X}(\,\bullet\,,\,\bullet\,)\colon \mathrm{D}^-(\mathscr{R}_X)^{\mathrm{op}} \times \mathrm{D}^+(\mathscr{R}_X) \to \mathrm{D}^+(\mathbf{k}_X),\\
&f^{-1}\colon \mathrm{D}^*(\mathscr{R}_Y) \to \mathrm{D}^*(f^{-1}\mathscr{R}_Y) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{R}f_*\colon \mathrm{D}^+(f^{-1}\mathscr{R}_Y)) \to \mathrm{D}^+(\mathscr{R}_Y),\\
&\mathrm{R}\Gamma(X;\,\bullet\,)\colon \mathrm{D}^+(\mathscr{R}_X) \to \mathrm{D}^+(\mathscr{R}(X)),\\
&\bullet \overset{\mathrm{L}}{\otimes}_{\mathscr{R}_X} \bullet\colon \mathrm{D}^-(\mathscr{R}_X^{\mathrm{op}}) \times \mathrm{D}^-(\mathscr{R}_X) \to \mathrm{D}^-(\mathbf{k}_X)\quad,\\
&\mathrm{j}_{U*} \simeq i_U^{-1}\colon \mathrm{D}^*(\mathscr{R}_X) \to \mathrm{D}^*(\mathscr{R}_U) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{j}_U^{-1} \simeq i_{U!}\colon \mathrm{D}^*(\mathscr{R}_U) \to \mathrm{D}^*(\mathscr{R}_X) \quad (* = \mathrm{b}, +, -),\\
&\mathrm{R}i_{U*}\colon \mathrm{D}^+(\mathscr{R}_U) \to \mathrm{D}^+(\mathscr{R}_X).
\end{aligned}
$$

One can also extend Theorem 9.3.3 to $\mathscr{R}$-modules. We leave the exact formulation to the reader who will be aware that formula (a)–(ii) needs to be modified if $\mathscr{R}_X$ is not commutative.

9.5 Ringed sites

Definition 9.5.1. (i) A $\mathbf{k}$-ringed site (or simply, a ringed site) $(X, \mathscr{O}_X)$ is a site X endowed with a sheaf $\mathscr{O}_X$ of *commutative* $\mathbf{k}_X$-algebras.

(ii) Let $(X, \mathscr{O}_X)$ be a ringed site. A locally free $\mathscr{O}_X$-module $\mathscr{M}$ of rank m is an $\mathscr{O}_X$-module such that there is a covering $\mathscr{S}$ of X and for each $V \in \mathscr{S}$, $\mathscr{O}_X$-linear isomorphisms $\mathscr{M}|_V \xrightarrow{\sim} (\mathscr{O}_X|_V)^m$.

(iii) If $m = 1$, one says that $\mathscr{M}$ is a line bundle.

(iv) If there exists a globally defined isomorphism $\mathscr{M} \xrightarrow{\sim} (\mathscr{O}_X)^m$ on X, one says that $\mathscr{M}$ is globally free.

We shall construct locally free sheaves.

Let $(X, \mathscr{O}_X)$ be a ringed site and consider the situation of Theorem 8.8.3. If all F_i's are sheaves of $\mathscr{O}_X|_{U_i}$ modules and the isomorphisms θ_{ji} are $\mathscr{O}_X|_{U_{ij}}$-linear, the sheaf F constructed in Theorem 8.8.3 will be naturally endowed with a structure of a sheaf of $\mathscr{O}_X$-modules.

Definition 9.5.2. Let $(X, \mathscr{O}_X)$ be a ringed site. Denote by $\mathscr{O}_X^\times$ the abelian sheaf of invertible sections of $\mathscr{O}_X$.

(i) A 1-cocycle $(\mathbf{c}, \mathscr{S})$ on X with values in $\mathscr{O}_X^\times$ is the data of a covering $\mathscr{S} = \{U_i\}_i$ of X and for each pair $U_i, U_j \in \mathscr{S}$ a section $c_{ij} \in \Gamma(U_{ij}; \mathscr{O}_X^\times)$, these data satisfying:

$$c_{ij} \cdot c_{jk} = c_{ik} \text{ on } U_{ijk}. \tag{9.5.1}$$

(ii) If there exists a family $c_i \in \Gamma(U_i; \mathscr{O}_X^\times)$ $(U_i \in \mathscr{S})$ such that

$$c_{ij} = c_i \cdot c_j^{-1} \text{ on } U_{ij}, \tag{9.5.2}$$

one says that the 1-cocycle $(\mathbf{c}, \mathscr{S})$ is a coboundary.

In the sequel, we still denote by c_{ij} the automorphism of $\mathscr{R}_X|_{U_{ji}}$ given by multiplication by the section c_{ij}.

Proposition 9.5.3. (i) *Consider a* 1*-cocycle* $(\mathbf{c}, \mathscr{S})$ *on* X *with values in* $\mathscr{R}_X^\times$. *There exists a unique sheaf* $\mathscr{L}_\mathbf{c}$ *with the following property: for each* $U_i \in \mathscr{S}$, *there exists an isomorphism* $\theta_i \colon \mathscr{R}_X|_{U_i} \xrightarrow{\sim} \mathscr{L}_\mathbf{c}|_{U_i}$ *and* $\theta_j^{-1} \circ \theta_i = c_{ij}$ *on* U_{ij} *for any* $U_i, U_j \in \mathscr{S}$.

(ii) *Assume that the* 1*-cocycle* $(\mathbf{c}, \mathscr{S})$ *in* (i) *is a coboundary. Then the sheaf* $\mathscr{L}_\mathbf{c}$ *is isomorphic to* $\mathscr{R}_X$.

Proof. (i) follows immediately from Theorem 8.8.3, by choosing $\theta_{ij} = c_{ji}$.
(ii) Consider the isomorphisms

$$\lambda_i := c_i \cdot \theta_i \colon \mathscr{R}_X|_{U_i} \xrightarrow{\sim} \mathscr{L}_\mathbf{c}|_{U_i}.$$

Since $\theta_{ij} = \theta_i \circ \theta_j^{-1}$ and $c_{ij} = c_i \cdot c_j^{-1}$, we get $\lambda_i|_{U_{ij}} = \lambda_j|_{U_{ij}}$. Applying Theorem 8.5.4 and Corollary 8.5.5, the isomorphisms λ_i's will glue as a global isomorphism $\mathscr{R}_X \xrightarrow{\sim} \mathscr{L}_\mathbf{c}$. □

Example 9.5.4. Let $X = \mathbb{P}^1(\mathbb{C})$, the Riemann sphere. Consider the covering of X by the two open sets $U_1 = \mathbb{C}$, $U_2 = X \setminus \{0\}$. One can glue $\mathscr{O}_X|_{U_1}$ and $\mathscr{O}_X|_{U_2}$ on $U_1 \cap U_2$ by using the isomorphism $f \mapsto z^m f$ $(k \in \mathbb{Z})$. One gets a locally free sheaf of rank one denoted by $\mathscr{O}_{\mathbb{P}^1}(m)$. For $m \neq 0$, this sheaf is not free.

Exercises to Chapter 9

Exercise 9.1. Let X be a topological space and let $\mathscr{U}$ be an open covering stable by finite intersections and which is a basis for the topology of X (that is, for any $x \in X$ and $V \in \mathrm{Op}_X$with $x \in V$, there exists $U \subset V$ with $x \in U$ and $U \in \mathscr{U}$). Let us consider $\mathscr{U}$ as a full subcategory of Op_X, hence as a presite. Denote by Y the presite $\mathscr{U}$ endowed with the Grothendieck topology defined as follows: a family of objects of $\mathscr{U}_U$ is a covering of U if it is a covering in X (that is, in the usual sense). Denote by $f \colon X \to Y$ the natural morphism of sites. Prove that $f_* \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_Y)$ is an equivalence of categories.

Exercise 9.2. Let $\mathbb{V}$ be a real finite dimensional vector space and let X be an open subset of $\mathbb{V}$. Let Y be the site defined as follows: the objects of $\mathscr{C}_Y$ are the non-empty convex open relatively compact subsets of X and the morphisms are the inclusions. The covering are those induced by the topology of X. Denote by $\rho\colon X \to Y$ the natural morphism of sites.

Using Exercise 9.1, show that ρ_* induces an equivalence $\mathrm{Mod}(\mathbf{k}_X) \xrightarrow{\sim} \mathrm{Mod}(\mathbf{k}_Y)$ and deduce that a sheaf F on X is constant if and only if for any morphism $u\colon U_1 \to U_2$ in Y, $F(u)$ is an isomorphism.

Exercise 9.3. Let $X = \mathbb{R}^2$ with coordinates (x, y), $Y = \mathbb{R}$ and let $p\colon X \to Y$ be the projection $(x, y) \mapsto x$. Let

$$\begin{aligned} &Z_0 = \{(x,y) \in \mathbb{R}^2; xy > 1\}, \quad Z_1 = \{(x,y) \in \mathbb{R}^2; xy \geq 1, x > 0\}, \\ &Z_2 = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 = 1\}. \end{aligned}$$

Calculate $\mathrm{R}p_*\mathbf{k}_{XZ_i}$ for $i = 0, 1, 2$.

Exercise 9.4. Let $X = \mathbb{R}^4$, $S = \{(x, y, z, t) \in \mathbb{R}^4; t^4 = x^2 + y^2 + z^2; t > 0\}$ and let $f\colon S \hookrightarrow X$ be the natural injection. Calculate $(\mathrm{R}f_*\mathbf{k}_S)_0$.

Exercise 9.5. Let $\mathscr{R}$ be a sheaf of commutative rings on a topological space X. Prove that a sheaf F of $\mathscr{R}$-modules is injective in the category $\mathrm{Mod}(\mathscr{R})$ if and only if, for any sheaf of ideals $\mathscr{I}$ of $\mathscr{R}$, the natural morphism $\Gamma(X; F) \to \mathrm{Hom}_{\mathscr{R}}(\mathscr{I}, F)$ is surjective.

Exercise 9.6. Let $\mathbb{P}^1(\mathbb{C})$ be the Riemann sphere.
(i) Construct an isomorphism of line bundles $\Omega^{(1)}_{\mathbb{P}^1(\mathbb{C})} \simeq \mathscr{O}_{\mathbb{P}^n(\mathbb{C})}(-2)$.
(ii) Prove that the line bundles $\mathscr{O}_{\mathbb{P}^1(\mathbb{C})}(l)$ and $\mathscr{O}_{\mathbb{P}^1(\mathbb{C})}(m)$ are not isomorphic for $l \neq m$. (Hint: reduce to the case where $l = 0$.)

Chapter 10

Sheaves on topological spaces

In this chapter we restrict our study of sheaves of $\mathbf{k}$-modules to the case of topological spaces. We first introduce the functors $(\,\bullet\,)_Z$ and $\Gamma_Z(\,\bullet\,)$ associated to a locally closed subset Z. Then we define Čech complexes associated with open or closed coverings and prove the "Leray's acyclic coverings theorem". Then we study locally constant abelian sheaves and prove that the cohomology of such sheaves is a homotopy invariant. Using the Čech complex associated to an open or a closed covering, we show how to compute the cohomology of spaces which admit coverings by contractible subsets. Finally, we apply these techniques to calculate the cohomology of some classical manifolds.

10.1 Restriction of sheaves

We shall identify a topological space X with the site associated to the category Op_X of open subsets of X, the morphisms being the inclusion morphisms and the coverings being the usual coverings: a small family $\{U_i\}_{i\in I}$ of open sets ia a covering of $U \in \mathrm{Op}_X$ if $U_i \subset U$ for all i and $\bigcup_i U_i = U$. Note that the site X has a terminal object, namely the whole space X.

We denote by

$$a_X\colon X \to \mathrm{pt} \tag{10.1.1}$$

the natural continuous map.

If A is a subset of a topological space X, we denote by $\overline{A}$ its closure, by $\mathrm{Int}A$ its interior and we set $\partial A = \overline{A} \setminus \mathrm{Int}A$.

Let Z be a subset of X, $i_Z\colon Z \hookrightarrow X$ the inclusion. One endows Z with the induced topology and for $F \in \mathrm{Mod}(\mathbf{k}_X)$, one sets:

$$\begin{aligned} F|_Z &= i_Z^{-1}F, \\ \Gamma(Z;F) &= \Gamma(Z;i_Z^{-1}F). \end{aligned}$$

If Z is open, these definitions agree with the previous ones. The morphism $F \to i_{Z*}i_Z^{-1}F$ defines the morphism $a_{X*}F \to a_{Z*}i_Z^{-1}F$, that is, the morphism:

$$\Gamma(X;F) \to \Gamma(Z;F).$$

One denotes by $s|_Z$ the image of a section s of F on X by this morphism.

Replacing X by an open subset U containing Z, we get the natural morphism:

$$\operatorname*{colim}_{U\supset Z}\Gamma(U;F)\to\Gamma(Z;F). \tag{10.1.2}$$

This morphism is injective. Indeed, if a section $s\in\Gamma(U;F)$ is zero in $\Gamma(Z;F)$, this implies $s_x=0$ for all $x\in Z$, hence $s=0$ on an open neighborhood of Z. But one shall take care that this morphism is not an isomorphism in general. This is true in some particular situations (see Proposition 10.1.2).

Definition 10.1.1. (i) A subset Z of a topological space X is relatively Hausdorff if two distinct points in Z admit disjoint neighborhoods in X. If $Z=X$, one says that X is Hausdorff.

(ii) A paracompact space X is a Hausdorff space such that for each open covering $\{U_i\}_{i\in I}$ of X there exists an open refinement $\{V_j\}_{j\in J}$ (*i.e.*, for each $j\in J$ there exists $i\in I$ such that $V_j\subset U_i$) which is locally finite.

Recall that, by its definition, a compact set is in particular Hausdorff.

If X is paracompact and $\{U_i\}_i$ is a locally finite open covering, there exists an open refinement $\{V_i\}_i$ such that $\overline{V_i}\subset U_i$. Closed subspaces of paracompact spaces are paracompact. Locally compact spaces countable at infinity (*i.e.*, countable union of compact subspaces), are paracompact.

Proposition 10.1.2. *Assume one of the following conditions:*

(i) *Z is open,*

(ii) *Z is a relatively Hausdorff compact subset of X,*

(iii) *Z is closed and X is paracompact.*

Then the morphism (10.1.2) is an isomorphism.

Proof. (i) is obvious.
(ii) Let $s\in\Gamma(Z;F)$. There exist a finite family of open subsets $\{U_i\}_{i=1}^n$ covering Z and sections $s_i\in\Gamma(U_i;F)$ such that $s_i|_{Z\cap U_i}=s|_{Z\cap U_i}$. Moreover, we may find another family of open sets $\{V_i\}_{i=1}^n$ covering Z such that $Z\cap\bar{V}_i\subset U_i$. We shall glue together the sections s_i on a neighnorhood of Z. For that purpose we may argue by induction on n and assume $n=2$. Set $Z_i=Z\cap\bar{V}_i$. Then $s_1|_{Z_1\cap Z_2}=s_2|_{Z_1\cap Z_2}$. Let W be an open subset of X such that $s_1|_W=s_2|_W$ and let $W_i(i=1,2)$ be an open subset of U_i such that $W_i\supset Z_i\setminus W$ and $W_1\cap W_2=\varnothing$. Such W_i's exist thanks to the hypotheses. Set $U'_i=W_i\cup W$, $(i=1,2)$. Then $s_1|_{U'_1\cap U'_2}=s_2|_{U'_1\cap U'_2}$. This defines $t\in\Gamma(U'_1\cup U'_2;F)$ with $t|_Z=s$.
(iii) We shall not give the proof here and refer to [God58]. ☐

Let $f\colon X\to Y$ be a continuous map and let F be a sheaf on X. Let $y\in Y$. The natural morphism $\operatorname*{colim}_{V\ni y}\Gamma(f^{-1}V;F)\to\Gamma(f^{-1}(y);F|_{f^{-1}(y)})$ defines the morphism:

$$(f_*F)_y\to\Gamma(f^{-1}(y);F|_{f^{-1}(y)}). \tag{10.1.3}$$

This morphism is not an isomorphism in general.

Examples 10.1.3. (i) Assume f is an open inclusion $U \hookrightarrow X$, let $G \in \mathrm{Mod}(\mathbf{k}_U)$ and choose $x \in \partial U$ $(= \bar{U} \setminus U)$. Then $f^{-1}(x) = \varnothing$ and $\Gamma(f^{-1}(x); G|_{f^{-1}(x)}) = 0$ but

$$(f_*G)_x = \operatorname*{colim}_V \Gamma(U \cap V; G),$$

where V ranges through the family of open neighborhoods of x in X, and this group is not zero in general.
(ii) Let $X = \mathbb{C}$ with coordinate $z = x+iy$, $Y = \mathbb{R}$, $f : X \to \mathbb{R}$ the map $f(x+iy) = y$. Then

$$\begin{aligned}
&(f_*\mathscr{O}_{\mathbb{C}})_0 \simeq \operatorname*{colim}_{\varepsilon>0} \Gamma(\{|y| < \varepsilon\}; \mathscr{O}_{\mathbb{C}}),\\
&\Gamma(f^{-1}(0); \mathscr{O}_{\mathbb{C}}|_{f^{-1}(0)}) \simeq \operatorname*{colim}_U \Gamma(U; \mathscr{O}_{\mathbb{C}}) \text{ where } U \text{ ranges through the family}\\
&\qquad\qquad\qquad\qquad\text{of open neighborhoods of } \mathbb{R} \text{ in } \mathbb{C}.
\end{aligned}$$

Note that the family of open neighborhoods of $\mathbb{R}$ in $\mathbb{C}$ is not countable.

10.2 Sheaves associated with a locally closed subset

Let X be a topological space, U an open subset of X and $F \in \mathrm{Mod}(\mathbf{k}_X)$. Recall that $F_U = \mathrm{j}_U^{-1}\,\mathrm{j}_{U*}F \simeq i_{U!}i_U^{-1}F$.

Propositions 10.2.1, 10.2.3 and 10.2.4 below are easy exercises whose proofs are left to the reader. Note that some of these results were already proved in Proposition 8.7.8.

Proposition 10.2.1. (i) *The functor* $(\bullet)_U \colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$, $F \mapsto F_U$, *is exact and commutes with inductive limits.*

(ii) *One has* $F_U \simeq F \otimes \mathbf{k}_{XU}$.

(iii) *For* $x \in X$, $(F_U)_x \simeq F_x$ *or* $(F_U)_x \simeq 0$ *according whether* $x \in U$ *or not.*

(iv) *Let* U_1 *and* U_2 *be two open subsets of* X. *Then* $(F_{U_1})_{U_2} = F_{U_1 \cap U_2}$.

(v) *Let* U_1 *and* U_2 *be two open subsets of* X. *Then there is an exact sequence*

$$0 \to F_{U_1\cap U_2} \xrightarrow{\alpha} F_{U_1} \oplus F_{U_2} \xrightarrow{\beta} F_{U_1\cup U_2} \to 0. \tag{10.2.1}$$

Here $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = \beta_1 - \beta_2$ *are induced by the natural morphisms* $\alpha_i : F_{U_1\cap U_2} \to F_{U_i}$ *and* $\beta_i \colon F_{U_i} \to F_{U_1\cup U_2}$.

Now set $S := X \setminus U$. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, define the sheaf F_S by

$$F_S = i_{S*}i_S^{-1}F. \tag{10.2.2}$$

Notation 10.2.2. For a closed set $S \subset X$ one sets $\mathbf{k}_{XS} := (\mathbf{k}_X)_S$. If there is no risk of confusion, we also write $\mathbf{k}_S$ instead of $\mathbf{k}_{XS}$. This last notation is justified by Remark 10.2.5 bellow.

Proposition 10.2.3. (i) *There is a natural exact sequence* $0 \to F_U \to F \to F_S \to 0$.

(ii) *The functor* $(\bullet)_S\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$, $F \mapsto F_S$, *is exact.*

(iii) *One has* $F_S \simeq F \otimes \mathbf{k}_{XS}$, *where one sets* $\mathbf{k}_{XS} := (\mathbf{k}_X)_S$ *for short.*

(iv) *For* $x \in X$, $(F_S)_x \simeq F_x$ *or* $(F_S)_x \simeq 0$ *according whether* $x \in S$ *or not.*

(v) *Let* S' *be another closed subset. Then* $(F_S)_{S'} = F_{S\cap S'}$.

(vi) *Let* S_1 *and* S_2 *be two closed subsets of* X. *Then the sequence below is exact:*

$$0 \to F_{S_1\cup S_2} \xrightarrow{\alpha} F_{S_1} \oplus F_{S_2} \xrightarrow{\beta} F_{S_1\cap S_2} \to 0. \tag{10.2.3}$$

Here $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = \beta_1 - \beta_2$ *are induced by the natural morphisms* $\alpha_i : F_{S_1\cup S_2} \to F_{S_i}$ *and* $\beta_i : F_{S_i} \to F_{S_1\cap S_2}$.

Recall that a locally closed set Z is the (non unique) intersection of an open subset U and a closed subset S of X. For $F \in \mathrm{Mod}(\mathbf{k}_X)$, one sets

$$\begin{aligned} &F_Z := (F_U)_S, \quad \mathbf{k}_{XZ} = (\mathbf{k}_X)_Z, \\ &\Gamma_Z(F) = \mathscr{H}om\,(\mathbf{k}_{XZ}, F), \quad \Gamma_Z(X;F) = \mathrm{Hom}\,(\mathbf{k}_{XZ}, F). \end{aligned} \tag{10.2.4}$$

One checks easily that the definition of F_Z does not depend on the choice of U and S such that $Z = U \cap S$.

Proposition 10.2.4. (i) *The functor* $(\bullet)_Z\colon \mathrm{Mod}(\mathbf{k}_X) \to \mathrm{Mod}(\mathbf{k}_X)$, $F \mapsto F_Z$, *is well defined and satisfies the properties* (ii)–(v) *of* Proposition 10.2.3 (*with* S *replaced by* Z).

(ii) *Let* Z *be as above and let* Z' *be a closed subset of* Z. *One has an exact sequence*

$$0 \to F_{Z\setminus Z'} \to F_Z \to F_{Z'} \to 0. \tag{10.2.5}$$

(iii) *One has for* $Z = U \cap S$:

$$\Gamma_Z(X;F) = \{s \in \Gamma(U;F); \mathrm{supp}(s) \text{ is contained in } Z\}.$$

Let U_1, U_2 be open subsets, S_1, S_2 closed subsets, Z a locally closed subset of X, Z' a closed subset of Z. Consider the exact sequences (10.2.1), (10.2.3) and (10.2.5). They give rise to distinguished triangles in the category $\mathrm{D}^+(\mathbf{k}_X)$:

$$\begin{aligned} &F_{U_1\cap U_2} \to F_{U_1} \oplus F_{U_2} \to F_{U_1\cup U_2} \xrightarrow{+1}, \\ &F_{S_1\cup S_2} \to F_{S_1} \oplus F_{S_2} \to F_{S_1\cap S_2} \xrightarrow{+1}, \\ &F_{Z\setminus Z'} \to F_Z \to F_{Z'} \xrightarrow{+1} . \end{aligned}$$

Choosing $F = \mathbf{k}_X$ and applying the functor $\mathrm{R}\mathscr{H}om\,(\bullet, F)$, we get new distinguished triangles, called Mayer-Vietoris triangles :

$$\mathrm{R}\Gamma_{U_1\cup U_2}F \to \mathrm{R}\Gamma_{U_1}F \oplus \mathrm{R}\Gamma_{U_2}F \to \mathrm{R}\Gamma_{U_1\cap U_2}F \xrightarrow{+1}, \tag{10.2.6}$$

$$\mathrm{R}\Gamma_{S_1\cap S_2}F \to \mathrm{R}\Gamma_{S_1}F \oplus \mathrm{R}\Gamma_{S_2}F \to \mathrm{R}\Gamma_{S_1\cup S_2}F \xrightarrow{+1}, \tag{10.2.7}$$

$$\mathrm{R}\Gamma_{Z'}(F) \to \mathrm{R}\Gamma_Z(F) \to \mathrm{R}\Gamma_{Z\setminus Z'}(F) \xrightarrow{+1} . \tag{10.2.8}$$

When applying $\mathrm{R}\Gamma(X;\bullet)$, we find other distinguished triangles and taking the cohomology, we find long exact sequences, such as for example the Mayer-Vietoris long exact sequences :

$$\begin{aligned}(10.2.9)\quad \cdots \to H^j(U_1\cup U_2;F) &\to H^j(U_1;F)\oplus H^j(U_2;F)\\ &\to H^j(U_1\cap U_2;F)\to H^{j+1}(U_1\cup U_2;F)\to\cdots\\ (10.2.10)\cdots \to H^j(X;F_{S_1\cup S_2}) &\to H^j(X;F_{S_1})\oplus H^j(X;F_{S_2})\\ &\to H^j(X;F_{S_1\cap S_2})\to H^{j+1}(X;F_{S_1\cup S_2})\to\cdots\end{aligned}$$

Remark 10.2.5. Let S be a closed subset of X. Then

$$\mathrm{R}\Gamma(X;F_S)\simeq \mathrm{R}\Gamma(S;F|_S). \tag{10.2.11}$$

Indeed, denote by $i_S\colon S\hookrightarrow X$ the embedding. Then

$$\begin{aligned}\mathrm{R}a_{X*}i_{S*}i_S^{-1}F &\simeq \mathrm{R}a_{X*}\mathrm{R}i_{S*}i_S^{-1}F\\ &\simeq \mathrm{R}a_X\circ i_{S*}i_S^{-1}F\simeq \mathrm{R}a_{S*}i_S^{-1}F.\end{aligned}$$

Note that (10.2.11) would not remain true when replacing S with an open subset.

When Z is locally closed in X, one also sets

$$H^j_Z(F)=H^j(\mathrm{R}\Gamma_Z(\bullet))(F),\quad H^j_Z(X;F)=H^j(\mathrm{R}\Gamma_Z(X;\bullet))(F). \tag{10.2.12}$$

10.3 Čech complexes for open coverings

The constructions and results of this section could be formulated in the more general setting of sites but for simplicity we reduce our study to topological spaces..

Let I_{ord} be a totally ordered set and denote by I the underlying set. For $J\subset I$, J finite, we denote by $|J|$ its cardinal. We endow J with the order induced by I.

Denote by $\{e_i\}_{i\in I}$ the canonical basis of $\mathbb{Z}^{\oplus I}$. For $J\subset I_{\mathrm{ord}}$, $J=\{i_0<i_1<\cdots<i_p\}$, we denote by e_J the element $e_{i_0}\wedge\cdots\wedge e_{i_p}$ of $\bigwedge^{p+1}\mathbb{Z}^{\oplus I}$. For σ a permutation of the set J with signature ε_σ, we have in $\bigwedge^{p+1}\mathbb{Z}^{\oplus I}$:

$$e_{\sigma(J)}=\varepsilon_\sigma e_J.$$

Consider a family $\mathscr{U}:=\{U_i\}_{i\in I}$ of open subsets of X. For $\varnothing\neq J\subset I$, J finite, set

$$U_J=\bigcap_{i\in J}U_i.$$

Now let $F\in\mathrm{Mod}(\mathbf{k}_X)$. For J as above and $i\in J$, we denote by $\beta_{(i,J)}\colon F_{U_J}\to F_{U_{J\setminus\{i\}}}$ the natural morphism. We set for $p\geq 0$

$$F^{\mathscr{U}}_p:=\bigoplus_{J\subset I_{\mathrm{ord}},|J|=p+1}F_{U_J}\otimes e_J$$

and we define the differential

$$d\colon F^{\mathscr{U}}_p\to F^{\mathscr{U}}_{p-1} \tag{10.3.1}$$

by setting for $s_J \in F_{U_J}$ (see Remark 5.7.8):

$$d(s_J \otimes e_J) = \sum_{i \in J} \beta_{(i,J)}(s_J) \otimes e_i \lfloor e_J.$$

One easily checks that

$$d \circ d = 0.$$

Then we have the Čech complex in $\mathrm{Mod}(\mathbf{k}_X)$ in which the term $F_p^{\mathscr{U}}$ is in degree $-p$.

$$F_\bullet^{\mathscr{U}} := \cdots \to F_p^{\mathscr{U}} \xrightarrow{d} \cdots \xrightarrow{d} F_1^{\mathscr{U}} \xrightarrow{d} F_0^{\mathscr{U}} \to 0. \tag{10.3.2}$$

We also consider the augmented complex

$$F_{\bullet,+}^{\mathscr{U}} := \cdots \to F_p^{\mathscr{U}} \xrightarrow{d} \cdots \xrightarrow{d} F_1^{\mathscr{U}} \xrightarrow{d} F_{\mathscr{U}}^0 \to F \to 0. \tag{10.3.3}$$

Proposition 10.3.1. *Let $\mathscr{U}$ be an open covering of X. Then for each $i_0 \in I$, the restriction to U_{i_0} of the complex (10.3.3) is homotopic to zero. In particular, $F_\bullet^{\mathscr{U}} \to F$ is a qis.*

Proof. Replacing all U_i by $U_i \cap U_{i_0}$, we may assume from the beginning that $X = U_{i_0}$. For each finite set $J \subset I$, $U_J = U_J \cap U_{i_0} = U_{\{i_0\} \cup J}$ and we get the identity morphism $\gamma_J \colon F_{U_J} \to F_{U_{\{i_0\} \cup J}}$. Define the homotopy $\lambda : F_p^{\mathscr{U}} \to F_{p+1}^{\mathscr{U}}$ by setting for $s_J \in F_{U_J}$,

$$\lambda(s_J \otimes e_J) = \gamma_J(s_J) \otimes e_{i_0} \wedge e_J.$$

Let us check the relation $d \circ \lambda + \lambda \circ d = \mathrm{id}$. Let $s_J \otimes e_J$ be a section of F_{U_J}. Then

$$d \circ \lambda(s_J \otimes e_J) = \sum_{i \in \{i_0\} \cup J} \beta_{(i,\{i_0\} \cup J)} \gamma_J(s_J) \otimes e_i \lfloor e_{i_0} \wedge e_J,$$
$$\lambda \circ d(s_J \otimes e_J) = \sum_{i \in J} \gamma_J \beta_{(i,J)}(s_J) \otimes e_{i_0} \wedge e_i \lfloor e_J.$$

We have $e_i \lfloor e_{i_0} \wedge e_J = -e_{i_0} \wedge e_i \lfloor e_J$. Hence, by summing these two relations, we find on the right hand side $\beta_{(i_0,J)} \gamma_J(s_J) \otimes e_{i_0} \lfloor e_{i_0} \wedge e_J = s_J \cdot e_J$. □

Example 10.3.2. Let $\{U_j\}_{j=0,1,2}$ be a covering of X. We get the exact complex:

$$0 \to F_{U_{012}} \xrightarrow{d_1} F_{U_{12}} \oplus F_{U_{02}} \oplus F_{U_{01}} \xrightarrow{d_0} F_{U_0} \oplus F_{U_1} \oplus F_{U_2} \xrightarrow{d_{-1}} F \to 0$$

with for example, $d_1(s_{012}) = \beta_{0,12}(s_{012}) - \beta_{1,02}(s_{012}) + \beta_{2,01}(s_{012})$.

Applying Proposition 10.3.1 with $F = \mathbf{k}_X$, we find the complex

$$\mathbf{k}_{X\bullet}^{\mathscr{U}} := \cdots \to (\mathbf{k}_X^{\mathscr{U}})_1 \xrightarrow{d} (\mathbf{k}_X^{\mathscr{U}})_0 \to 0 \tag{10.3.4}$$

and we have the isomorphism

$$F_\bullet^{\mathscr{U}} \simeq \mathbf{k}_{X\bullet}^{\mathscr{U}} \otimes F.$$

It is then natural to consider the complex of sheaves

$$\mathscr{C}^\bullet(\mathscr{U}, F) := \mathscr{H}om\,(\mathbf{k}_{X\bullet}^{\mathscr{U}}, F) \tag{10.3.5}$$

and the morphism

$$F \to \mathscr{C}^\bullet(\mathscr{U}, F), \tag{10.3.6}$$

that is, the complex

$$0 \to F \to \mathscr{C}^0(\mathscr{U}, F) \xrightarrow{d} \mathscr{C}^1(\mathscr{U}, F) \to \cdots ,$$

where

$$\mathscr{C}^p(\mathscr{U}, F) = \prod_{|J|=p+1} \Gamma_{U_J}(F) \otimes e_J \text{ with } \mathscr{C}^{-1}(\mathscr{U}, F) = F,$$

and the morphisms: $d^p\colon \mathscr{C}^p(\mathscr{U}, F) \to \mathscr{C}^{p+1}(\mathscr{U}, F)$ $(p \geq -1)$ are defined by

$$d^p(s_J \otimes e_J) = \sum_{i \in I} \alpha_{(i,J)}(s_J) \otimes e_i \wedge e_J,$$

where $\alpha_{(i,J)}$ is the natural morphism $\Gamma_{U_J}(F) \to \Gamma_{U_{J \cup \{i\}}}(F)$. Moreover, $d^{-1}\colon F \to \prod_i \Gamma_{U_i}(F)$ is the natural morphism.

Proposition 10.3.3. *Assume that $\mathscr{U}$ is an open covering of X. The sequence of sheaves (10.3.6) is exact.*

Proof. It is enough to check that this sequence is exact on each $U \in \mathscr{U}$. The additive functor $\mathscr{H}om(\cdot, F)$ sends a complex homotopic to zero to a complex homotopic to zero. Applying this functor to the complex (10.3.3) in which one chooses $F = \mathbf{k}_X$, the result follows from Proposition 10.3.1. ☐

Theorem 10.3.4. (The Leray's acyclic coverings theorem.) *Let $\mathscr{U} = \{U_i\}_{i \in I}$ be an open covering of X, let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and assume that for any finite subset J of I and any $p > 0$, one has $H^p(U_J; F) = 0$. Then $\mathrm{R}\Gamma(X; \mathscr{C}^\bullet(\mathscr{U}, F)) \simeq \mathrm{R}\Gamma(X; F)$.*

Proof. Let $F^\bullet$ be an injective resolution of F and consider the double complex:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Gamma(X; F) \arrow[r] \arrow[d] & \Gamma(X; \mathscr{C}^\bullet(\mathscr{U}, F)) \arrow[d] \\
0 \arrow[r] & \Gamma(X; F^\bullet) \arrow[r] & \Gamma(X; \mathscr{C}^\bullet(\mathscr{U}, F^\bullet))
\end{tikzcd}$$

(i) All rows, except the first one, are exact. In fact, they are obtained by applying the functor $\Gamma(X; \cdot)$ to the exact complex of injective sheaves $0 \to F^j \to \mathscr{C}^\bullet(\mathscr{U}, F^j)$.

(ii) All columns, except the first one, are exact. In fact, it is enough to prove that for $J \subset I$, J finite, the complex $0 \to \Gamma(X; \Gamma_{U_J}(F)) \to \Gamma(X; \Gamma_{U_J}(F^\bullet))$ is exact. This complex is isomorphic to the complex $0 \to \Gamma(U_J; F) \to \Gamma(U_J; F^\bullet)$ which is exact by the hypothesis.

(iii) Applying Theorem 5.6.4, one deduces from (i) and (ii) that the cohomology of the first row is isomorphic to that of the first column. ☐

Example 10.3.5. Let $\{U_j\}_{j=0,1,2}$ be an open covering of X, and assume that $H^p(U_J;F)=0$ for all $p>0$ and all $J\subset\{0,1,2\}$. Then $H^j(X;F)$ is isomorphic to the j-th cohomology group of the complex

$$0\to\bigoplus_{j=0,1,2}\Gamma(U_j;F)\xrightarrow{d^0}\bigoplus_{J=01,12,02}\Gamma(U_J;F)\xrightarrow{d^1}\Gamma(U_{012};F)\to 0$$

where the d^j's are linear combinations of the restriction morphisms affected with the sign ±. For example, $d^1|_{\Gamma(U_{02};F)}$ is affected with the sign $-$.

10.4 Čech complexes for closed coverings

We shall adapt the construction of §10.3 to the case of closed coverings.

As in § 10.3, we consider a total ordered set I_{ord} and denote by I the underlying set. We keep the notations of this section.

Let $\mathscr{S}=\{S_i\}_{i\in I}$ be a family of closed subsets of X and let $F\in\mathrm{Mod}(\mathbf{k}_X)$. For $J\subset I$ we set

$$\begin{aligned}&S_J:=\textstyle\bigcap_{j\in J}S_j,\quad S_{-1}=\textstyle\bigcup_{i\in I}S_i,\\&F^p_{\mathscr{S}}:=\oplus_{|J|=p+1}F_{S_J},\quad F^{-1}_{\mathscr{S}}=F_S.\end{aligned}\tag{10.4.1}$$

For $J\subset I$ and $i\in I$, we denote by $\alpha_{(i,J)}\colon F_{S_J}\to F_{S_{J\cup\{i\}}}$ the natural restriction morphism. We set for $p\geq 0$

$$F^p_{\mathscr{S}}:=\bigoplus_{J\subset I_{\mathrm{ord}},|J|=p+1}F_{S_J}\otimes e_J$$

and we define the differential

$$d\colon F^p_{\mathscr{S}}\to F^{p+1}_{\mathscr{S}}\tag{10.4.2}$$

by setting for $s_J\in F_{S_J}$:

$$d(s_J\otimes e_J)=\sum_{i\in I}\alpha_{(i,J)}(s_J)\otimes e_i\wedge e_J.$$

One easily checks that

$$d\circ d=0$$

and we obtain a complex

$$F^\bullet_{\mathscr{S}}:=0\to F^0_{\mathscr{S}}\xrightarrow{d^0}F^1_{\mathscr{S}}\xrightarrow{d^1}\cdots.\tag{10.4.3}$$

We also consider the augmented complex

$$F^{\bullet,+}_{\mathscr{S}}:=0\to F_S\xrightarrow{d^{-1}}F^0_{\mathscr{S}}\xrightarrow{d^0}F^1_{\mathscr{S}}\xrightarrow{d^1}\cdots.\tag{10.4.4}$$

Proposition 10.4.1. *Consider a family $\mathscr{S}=\{S_i\}_{i\in I}$ of closed subsets of X. Then the complex (10.4.4) is exact. Equivalently, $F_S\to F^\bullet_{\mathscr{S}}$ is a qis.*

Proof. Let $x \in X$ and denote by $M_x^\bullet := (F_{\mathscr{S}}^{\bullet,+})_x$ the stalk of the complex (10.4.4) at x. It is enough to check that this complex is exact. Let $K = \{i \in I; x \in S_i\}$. Replacing I with K, we may assume from the beginning that $x \in S_i$ for all $i \in I$. In this case, $M_x^\bullet$ is the Koszul complex associated with the module $M = F_x$ and the family of morphisms $\{\varphi_i\}_{i\in I}$ with $\varphi_i = \mathrm{id}_M$ for all $i \in I$. This last complex is clearly exact. □

Corollary 10.4.2. *Consider a finite closed covering* $\mathscr{S} = \{S_i\}_{i\in I}$ *of* X. *Let* $F \in \mathrm{Mod}(\mathbf{k}_X)$ *and assume that for any subset* J *of* I *and any* $p > 0$, *one has* $H^p(X; F_{S_J}) = 0$. *Then* $\mathrm{R}\Gamma(X; F) \simeq \mathrm{R}\Gamma(X; F_{\mathscr{S}}^\bullet)$.

Example 10.4.3. Assume that $X = S_0 \cup S_1 \cup S_2$, where the S_i's are closed subsets. We get the exact complex of sheaves

$$0 \to F \xrightarrow{d^{-1}} F_{S_0} \oplus F_{S_1} \oplus F_{S_2} \xrightarrow{d^0} F_{S_{12}} \oplus F_{S_{02}} \oplus F_{S_{01}} \xrightarrow{d^1} F_{S_{012}} \to 0.$$

Let us denote by

$$s_i\colon F \to F_{S_i}, s_{ij}^a\colon F_{S_a} \to F_{S_{ij}}, s_{\widehat{\mathbf{k}}}\colon F_{S_{ij}} \to F_{S_{012}}\,(a,i,j,k) \in \{0,1,2\}),$$

the natural morphisms. Then

$$d^{-1} = \begin{pmatrix} s_0, \\ s_1, \\ s_2 \end{pmatrix}, d^0 = \begin{pmatrix} 0 & -s_{12}^1 & s_{12}^2 \\ -s_{02}^0 & 0 & s_{02}^2 \\ -s_{01}^0 & s_{01}^1 & 0 \end{pmatrix} d^1 = (s_{\widehat{2}}, -s_{\widehat{0}}, s_{\widehat{1}}).$$

10.5 Flabby sheaves

Definition 10.5.1. On a topological space X, an object $F \in \mathrm{Mod}(\mathbf{k}_X)$ is flabby if for any open subset U of X the restriction map $\Gamma(X; F) \to \Gamma(U; F)$ is surjective.

By applying the functor $\mathrm{Hom}\,(\,\bullet\,, F)$ to the epimorphism $\mathbf{k}_X \to \mathbf{k}_{XU}$, one sees that injective sheaves are flabby. The converse is true if $\mathbf{k}$ is a field (see Exercise 9.5).

Proposition 10.5.2. *Let* F *be a flabby sheaf on* X.

(i) *If* U *is open in* X, $F|_U$ *is flabby on* U,

(ii) *if* $f\colon X \to Y$ *is a continuous map,* f_*F *is flabby on* Y,

(iii) *if* Z *be a locally closed subset of* X, $\Gamma_Z(F)$ *is flabby.*

Proof. (i)–(ii) are obvious.
(iii) Let U be an open subset containing Z as a closed subset and let V be an open subset of X. Since

$$\begin{aligned} \Gamma(V; \Gamma_Z(F)) &\simeq \Gamma(U \cap V; \Gamma_Z(F)) \\ &\simeq \Gamma_{Z\cap V}(U \cap V; F), \end{aligned}$$

we may assume from the beginning (replacing X by U) that Z is closed in X. Let $s \in \Gamma_{Z\cap V}(V; F)$ and consider the section $0 \in \Gamma(X \setminus Z; F)$. These two sections coincide on $V \cap (X \setminus Z)$, hence define a section $\widetilde{s} \in \Gamma(V \cup (X \setminus Z); F$. There exists $t \in \Gamma(X; F)$ whose restriction to $V \cup (X \setminus Z$ is $\widetilde{s}$. Therefore, $t|_{X\setminus Z} = 0$ and $\mathrm{supp}(t) \subset Z$. □

Proposition 10.5.3. *Let* $0 \to F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \to 0$ *be an exact sequence of sheaves, and assume* F' *is flabby. Then the sequence*

$$0 \to \Gamma(X; F') \xrightarrow{\alpha} \Gamma(X; F) \xrightarrow{\beta} \Gamma(X; F'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma(X; F'')$ and let $\sigma = \{(U; s);\ U \text{ open in } X,\ s \in \Gamma(U; F),\ \beta(s) = s''|_U\}$. Then σ is naturally inductively ordered. Let $(U; s)$ be a maximal element, and assume $U \neq X$.

Let $x \in X \setminus U$, let V be an open neighborhood of x and let $t \in \Gamma(U; F)$ such that $\beta(t) = s''|_V$. Such a pair $(V; t)$ exists since $\beta : F_x \to F''_x$ is surjective. On $U \cap V$, $s - t \in \Gamma(U \cap V; F')$. Let $r \in \Gamma(X; F')$ which extends $s - t$. Then $s - (t + r) = 0$ on $U \cap V$, hence there exists a section $\tilde{s} \in \Gamma(U \cup V; F)$ with $\tilde{s}|_U = s$, $\tilde{s}|_V = t + r$, and $\beta(\tilde{s}) = s''$. This is a contradiction. ☐

Proposition 10.5.4. *Let* $0 \to F' \to F \to F'' \to 0$ *be an exact sequence of sheaves. Assume* F' *and* F *are flabby. Then* F'' *is flabby.*

Proof. Let U be an open subset of X and consider the diagram:

$$\begin{tikzcd}
\Gamma(X; F) \arrow[r] \arrow[d, "\alpha"] & \Gamma(X; F'') \arrow[r] \arrow[d, "\gamma"] & 0 \\
\Gamma(U; F) \arrow[r, "\beta"] & \Gamma(U; F'') \arrow[r] & 0
\end{tikzcd}$$

Then α is surjective since F is flabby and β is surjective since F' is flabby, in view of the preceding proposition. This implies γ is surjective, hence F'' is flabby. ☐

Theorem 10.5.5. *The category of flabby sheaves is injective with respect to the functors* $\Gamma(X; \cdot)$, $\Gamma_Z(\cdot)$, f_*.

Proof. Since the category of sheaves has enough injectives, and injective sheaves are flabby, the result for $\Gamma(X; \cdot)$ follows from Propositions 10.5.3 and 10.5.4, and the other functors are similarly treated. ☐

Proposition 10.5.6. *Let* $X = \bigcup_{i \in I} U_i$ *be an open covering of* X *and let* $F \in \mathrm{Mod}(\mathbf{k}_X)$. *Assume that* $F|_{U_i}$ *is flabby for all* $i \in I$. *Then* F *is flabby.*

In other words, to be flabby is a local property.

Proof. Let U be an open subset of X and let $s \in F(U)$. Let us prove that s extends to a global section of F. Let $\mathfrak{S}$ be the family of pairs (t, V) such that V is open and contains U and $t|_U = s$. We order $\mathfrak{S}$ as follows:$(t, V) \leq (t', V')$ if $V \subset V'$ and $t'|_V = t$. Then $\mathfrak{S}$ is inductively ordered. Therefore, there exists a maximal element (t, V). Let us show that $V = X$. Otherwise, there exists $x \in X \setminus V$ and an $i \in I$ such that $x \in U_i$. Then $t|_{U_i \cap V} \in F(U_i \cap V)$ extends to a section $t_i \in F(U_i)$. Since $t_i|_{U_i \cap V} = t|_{U_i \cap V}$, the section t extends to a section on $V \cup U_i$ which contradicts the fact that V is maximal. ☐

By a similar argument, one proves (see [KS90, (2.4.1)]:

Proposition 10.5.7. *A sheaf F on X is flabby if and only if for each open set U, the morphism $F \to \Gamma_U F$ is an epimorphism.*

Proposition 10.5.8. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then F is injective if and only if the functor $\mathscr{H}om\,(\bullet\,,\,F)$ is exact.*

Proof. (i) We already know that the condition is necessary by Proposition 9.2.10.
(ii) Let us prove that the condition is sufficient. Since $\mathrm{Hom}\,(G,F) \simeq \Gamma(X;\mathscr{H}om\,(G,F))$, applying Proposition 10.5.3, it is enough to prove that for any $G \in \mathrm{Mod}(\mathbf{k}_X)$, the sheaf $\mathscr{H}om\,(G,F)$ is flabby. For U open, the morphism $G_U \to G$ is a monomorphism and thus

$$\mathscr{H}om\,(G,F) \to \mathscr{H}om\,(G_U,F) \simeq \Gamma_U \mathscr{H}om\,(G,F)$$

is an epimorphism. Then apply Proposition 10.5.7. □

Corollary 10.5.9. *Let $X = \bigcup_{i\in I} U_i$ be an open covering of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Assume that $F|_{U_i}$ is injective for all $i \in I$. Then F is injective.*

In other words, to be injective is a local property.

Definition 10.5.10. Let $d \in \mathbb{N}$. One says that X has flabby dimension $\leq d$ if any $F \in \mathrm{Mod}(\mathbf{k}_X)$ admits a resolution of length $\leq d$ by flabby sheaves. One says that X has finite flabby dimension if there exists $d \geq 0$ such that X has flabby dimension $\leq d$. (See Definition **??**.)

10.6 Sheaves on the interval $[0,1]$

Lemma 10.6.1. *Let $I = [0,1]$ and let $F \in \mathrm{Mod}(\mathbf{k}_I)$. Then:*

(i) *For $j > 1$, one has $H^j(I;F) = 0$.*

(ii) *If $F(I) \to F_t$ is an epimorphism for all $t \in I$, then $H^1(I;F) \simeq 0$.*

Proof. Let $j \geq 1$ and let $s \in H^j(I;F)$. For $0 \leq t_1 \leq t_2 \leq 1$, consider the morphism:

$$f_{t_1,t_2} \colon H^j(I;F) \to H^j([t_1,t_2];F)$$

and let

$$J = \{t \in [0,1];\ f_{0,t}(s) = 0\}.$$

Since $H^j(\{0\};F) = 0$ for $j \geq 1$, we have $0 \in J$. Since $f_{0,t}(s) = 0$ implies $f_{0,t'}(s) = 0$ for $0 \leq t' \leq t$, J is an interval. Since $H^j([0,t_0];F) = \operatorname*{colim}_{t>t_0} H^j([0,t];F)$ (see 10.15), this interval is open. It remains to prove that J is closed. For $0 \leq t \leq t_0$, consider the Mayer-Vietoris sequence (see (10.2.10) and (10.2.11)):

$$\cdots \to H^j([0,t_0];F) \to H^j([0,t];F) \oplus H^j([t,t_0];F) \to H^j(\{t\};F) \to \cdots$$

For $j > 1$, or else for $j = 1$ assuming $H^0(I;F) \to H^0(\{t\};F)$ is surjective, we obtain:

$$H^j([0,t_0];F) \simeq H^j([0,t];F) \oplus H^j([t,t_0];F). \tag{10.6.1}$$

Let $t_0 = \sup\,\{t; t \in J\}$. Then $f_{0,t}(s) = 0$, for all $t < t_0$. On the other hand,

$$\operatorname*{colim}_{t<t_0} H^j([t,t_0];F) = 0.$$

Hence, there exists $t < t_0$ with $f_{t,t_0}(s) = 0$. By (10.6.1), this implies $f_{0,t_0}(s) = 0$. Hence $t_0 \in J$. ☐

Lemma 10.6.2. *let $X = U_1 \cup U_2$ be a covering of the topological space X by two open sets. Let F be a sheaf on X and assume that:*

(i) *$U_{12} = U_1 \cap U_2$ is connected and non empty,*

(ii) *$F|_{U_i}$ $(i = 1,2)$ is a constant sheaf.*

Then F is a constant sheaf.

Proof. It follows from the hypothesis that there is a $\mathbf{k}$-module M and isomorphisms $\theta_i \colon F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$. Let $\theta_{12} = \theta_1 \circ \theta_2^{-1} \colon (M_X)|_{U_1\cap U_2} \xrightarrow{\sim} (M_X)|_{U_1\cap U_2}$. Since $U_1 \cap U_2$ is connected and non empty, $\Gamma(U_1 \cap U_2; \mathscr{H}om\,(M_X, M_X)) \simeq \mathrm{Hom}\,(M, M)$ and θ_{12} defines an invertible element of $\mathrm{Hom}\,(M,M)$. Using the map $\mathrm{Hom}\,(M,M) \to \Gamma(X; \mathscr{H}om\,(M_X, M_X))$, we find that θ_{12} extends as an isomorphism $\theta : M_X \simeq M_X$ all over X. Now define the isomorphisms: $\alpha_i \colon F|_{U_i} \xrightarrow{\sim} (M_X)|_{U_i}$ by $\alpha_1 = \theta_1$ and $\alpha_2 = \theta|_{U_2} \circ \theta_2$. Then α_1 and α_2 will glue together to define an isomorphism $F \xrightarrow{\sim} M_X$. ☐

Proposition 10.6.3. *Let I denote the interval* $[0,1]$.

(i) *Let F be a locally constant sheaf on I. Then F is a constant sheaf.*

(ii) *In particular, if $t \in I$, the morphism $\Gamma(I;F) \to F_t$ is an isomorphism.*

(iii) *Moreover, if $F = M_I$ for a $\mathbf{k}$-module M, then the composition*

$$M \simeq F_0 \xleftarrow{\sim} \Gamma(I; M_I) \xrightarrow{\sim} F_1 \simeq M$$

is the identity of M.

Proof. (i) We may find a finite open covering $U_i, (i = 1,\dots,n)$ such that F is constant on U_i, $U_i \cap U_{i+1}$ $(1 \le i < n)$ is non empty and connected and $U_i \cap U_j = \varnothing$ for $|i-j| > 1$. By induction, we may assume that $n = 2$. Then the result follows from Lemma 10.6.2.

(ii)–(iii) are obvious. ☐

10.7 Invariance by homotopy

In this section, we shall prove that the cohomology of locally constant sheaves is a homotopy invariant. First, we define what it means.

In the sequel, we denote by I the closed interval $I = [0,1]$.

Definition 10.7.1. Let X and Y be two topological spaces.

(i) Let f_0 and f_1 be two continuous maps from X to Y. One says that f_0 and f_1 are homotopic if there exists a continuous map $h\colon I \times X \to Y$ such that $h(0,\cdot) = f_0$ and $h(1,\cdot) = f_1$.

(ii) Let $f\colon X \to Y$ be a continuous map. One says that f is a homotopy equivalence if there exists $g\colon Y \to X$ such that $f \circ g$ is homotopic to id_Y and $g \circ f$ is homotopic to id_X. In such a case one says that X and Y are homotopic.

(iii) One says that a topological space X is contractible if X is homotopic to a point $\{x_0\}$.

One checks easily that the relation "f_0 is homotopic to f_1" is an equivalence relation. If $f_0, f_1\colon X \rightrightarrows Y$ are homotopic, one gets the diagram

(10.7.1)
$$\begin{tikzcd}
X \simeq \{t\} \times X \arrow[r, hook, "i_t"] \arrow[rr, bend left, "f_t"] & I \times X \arrow[r, "h"] \arrow[d, "p"] & Y \\
& X &
\end{tikzcd}$$

where $t \in I$, $i_t\colon X \simeq \{t\} \times X \hookrightarrow I \times X$ is the embedding, p is the projection and $f_t = h \circ i_t$.

A topological space is contractible if and only if there exist $g\colon \{x_0\} \to X$ and $f\colon X \to \{x_0\}$ such that $f \circ g$ is homotopic to id_X. Replacing x_0 with $g(x_0)$, this means that there exists $h\colon I \times X \to X$ such that $h(0,x) = \mathrm{id}_X$ and $h(1,x)$ is the map $x \mapsto x_0$. Note that a contractible space is non-empty.

Example 10.7.2. Let V be a real vector space. A non empty convex set in V as well as a closed cone are contractible sets.

Let $f\colon X \to Y$ be a continuous map and let $G \in \mathrm{Mod}(\mathbf{k}_Y)$. Remark that $a_X \simeq a_Y \circ f$. The morphism of functors $\mathrm{id} \to \mathrm{R}f_* \circ f^{-1}$ defines the morphism $\mathrm{R}a_{Y*} \to \mathrm{R}a_{Y*} \circ \mathrm{R}f_* \circ f^{-1} \simeq \mathrm{R}a_{X*} \circ f^{-1}$. We get the morphism:

$$f^\sharp\colon \mathrm{R}\Gamma(Y;G) \to \mathrm{R}\Gamma(X;f^{-1}G). \tag{10.7.2}$$

If $g\colon Y \to Z$ is another morphism, we have:

$$f^\sharp \circ g^\sharp = (g \circ f)^\sharp. \tag{10.7.3}$$

The aim of this section is to prove:

Theorem 10.7.3. (Invariance by homotopy Theorem.) *Let $f_0, f_1\colon X \rightrightarrows Y$ be two homotopic maps and let G be a locally constant sheaf on Y. Consider the two morphisms $f_t^\sharp\colon \mathrm{R}\Gamma(Y;G) \to \mathrm{R}\Gamma(X;f_t^{-1}G)$, for $t = 0,1$. Then there exists an isomorphism $\theta\colon \mathrm{R}\Gamma(X;f_0^{-1}G) \to \mathrm{R}\Gamma(X;f_1^{-1}G)$ such that $\theta \circ f_0^\sharp = f_1^\sharp$.*

If $G = M_Y$ for some $M \in \mathrm{Mod}(\mathbf{k})$, then, identifying $f_t^{-1}M_Y$ with M_X $(t = 0,1)$, we have $f_1^\sharp = f_0^\sharp$.

This is visualized by the diagram

$$\begin{tikzcd}
& \mathrm{R}\Gamma(Y;G) \arrow[dl, "f_0^\sharp"'] \arrow[dr, "f_1^\sharp"] & \\
\mathrm{R}\Gamma(X;f_0^{-1}G) \arrow[rr, "\theta", "\sim"'] & & \mathrm{R}\Gamma(X;f_1^{-1}G).
\end{tikzcd}$$

Proof of the main theorem

In order to prove Theorem 10.7.3, we need several lemmas.

Recall that the maps $p\colon I\times X\to X$ and $i_t\colon X\to I\times X$ are defined in (10.7.1). We also introduce the notation $I_x := I\times\{x\}$.

Lemma 10.7.4. *Let* $F\in\operatorname{Mod}(\mathbf{k}_X)$. *Then*

(i) $F\xrightarrow{\sim}\mathrm{R}p_*p^{-1}F$,

(ii) *the morphism* $p^\sharp\colon \mathrm{R}\Gamma(X;F)\to\mathrm{R}\Gamma(I\times X;p^{-1}F)$ *is an isomorphism,*

(iii) *the morphisms* $i_t^\sharp\colon \mathrm{R}\Gamma(I\times X;p^{-1}F)\to\mathrm{R}\Gamma(X;F)$ *are isomorphisms and do not depend on* $t\in I$.

Proof. (i) Let $x\in X$ and let $t\in I$. Since I is compact, a neighborhood system of $I\times\{x\}$ in $I\times X$ is given by the family $I\times V$ where V ranges trhough the family of open neighborhoods of x in X.Therefore, one gets the isomorphism $((\mathrm{R}p_*)p^{-1}F)_x\simeq\mathrm{R}\Gamma(I_x;p^{-1}F|_{I_x})$. This complex is concentrated in degree 0 and is isomorphic to F_x by Lemma 10.6.1.

(ii) We have $\mathrm{R}a_{X*}\mathrm{R}p_*p^{-1}F\simeq\mathrm{R}a_{X*}F$ by (i). Hence $p^\sharp$ is an isomorphism.

(iii) By (10.7.3), $i_t^\sharp\circ p^\sharp$ is the identity, and $p^\sharp$ is an isomorphism by (i). Hence, $i_t^\sharp$ which is the inverse of $p^\sharp$ does not depend on t. ☐

Lemma 10.7.5. *Let* $H\in\operatorname{Mod}(\mathbf{k}_{I\times X})$ *be a locally constant sheaf. Then*

(i) *the natural morphism* $p^{-1}\mathrm{R}p_*H\to H$ *is an isomorphism,*

(ii) *for each* $t\in I$, *the morphism* $i_t^\sharp\colon\mathrm{R}\Gamma(I\times X;H)\to\mathrm{R}\Gamma(X;i_t^{-1}H)$ *is an isomorphism.*

(iii) *If* $H=M_{I\times X}$ *for some* $M\in\operatorname{Mod}(\mathbf{k})$, *the isomorphism* $i_t^\sharp\colon\mathrm{R}\Gamma(I\times X;M_{I\times X})\to\mathrm{R}\Gamma(X;M_X)$ *does not depend on* t.

Proof. (i) One has

$$\begin{aligned}(p^{-1}\mathrm{R}p_*H)_{(t,x)}&\simeq(\mathrm{R}p_*H)_x\\&\simeq\mathrm{R}\Gamma(I_x;H|_{I_x})\simeq H_{(t,x)}.\end{aligned}$$

Here the last isomorphism follows from Lemmas 10.6.3 and 10.6.1.

(ii) Consider the commutative diagram

$$\begin{tikzcd}\mathrm{R}\Gamma(I\times X;H) \arrow[d, "i_t^\sharp"'] & \mathrm{R}\Gamma(I\times X;p^{-1}\mathrm{R}p_*H) \arrow[l, "\sim"'] \arrow[d, "i_t^\sharp"'] \\ \mathrm{R}\Gamma(X;i_t^{-1}H) & \mathrm{R}\Gamma(X;i_t^{-1}p^{-1}\mathrm{R}p_*H) \arrow[l, "\sim"']\end{tikzcd}$$

The horizontal arrows are isomorphisms by (i) and the vertical arrow on the right is an isomorphism by Lemma 10.7.4 (ii).

(iii) follows from Lemma 10.7.4 (iii). ☐

End of the proof of Theorem 10.7.3. Set $H=h^{-1}G$. Then $F_t=i_t^{-1}H$ and the results follow from Lemma 10.7.5 (ii)–(iii). ☐

Corollary 10.7.6. *Assume* $f\colon X \to Y$ *is a homotopy equivalence and let* G *be a locally constant sheaf on* Y*. Then* $\mathrm{R}\Gamma(X, f^{-1}G) \simeq \mathrm{R}\Gamma(Y; G)$*.*

In other words, the cohomology of locally constant sheaves on topological spaces is a homotopy invariant.

Proof. Let $g\colon Y \to X$ be a map such that $f \circ g$ and $g \circ f$ are homotopic to the identity of Y and X, respectively. Consider $f^\sharp\colon \mathrm{R}\Gamma(Y; G) \to \mathrm{R}\Gamma(X; f^{-1}G)$ and $g^\sharp\colon \mathrm{R}\Gamma(X; f^{-1}G) \to \mathrm{R}\Gamma(Y; G)$. Then: $(f \circ g)^\sharp = g^\sharp \circ f^\sharp \simeq \mathrm{id}_X^\sharp = \mathrm{id}$ and $(g \circ f)^\sharp = f^{\sharp j} \circ g^\sharp \simeq \mathrm{id}_Y^\sharp = \mathrm{id}$. ☐

Corollary 10.7.7. *If* X *is contractible and* $M \in \mathrm{Mod}(\mathbf{k})$*, then* $\mathrm{R}\Gamma(X; M_X) \simeq M$*.*

Proposition 10.7.8. *Assume* X *is contractible and* $F \in \mathrm{Mod}(\mathbf{k}_X)$ *is locally constant. Then* F *is constant.*

Proof. Set $M = \Gamma(X; F)$. The morphism $a_X^{-1} a_{X*} F \to F$ defines the morphism $M_X \to F$. Let us show it is an isomorphism. Let $x_0 \in X$. The map $X \to x_0$ is a homotopy equivalence. Applying Corollary 10.7.7 to the map $\iota\colon \{x_0\} \hookrightarrow X$, we get the isomorphism $F_{x_0} \simeq \Gamma(X; F) \simeq (M_X)_{x_0}$. ☐

We shall apply Theorem 10.7.3 to calculate the cohomology of various spaces. Recall (10.4.1).

Proposition 10.7.9. *Let* $\mathscr{S} = \{S_i\}_{i \in I}$ *be a finite closed covering of* X*. Assume for each non empty subset* $J \subset I$*,* S_J *is contractible or empty. Let* F *be a locally constant sheaf on* X*. Then* $H^j(X; F)$ *is isomorphic to the* j*-th cohomology object of the complex* $\Gamma(X; F^\bullet_{\mathscr{S}})$*.*

Proof. Recall that if Z is closed in X, then $\Gamma(X; F_Z) \simeq \Gamma(Z; F|_Z)$. By Corollary 10.7.7 and Proposition **??**, the sheaves $F^p_{\mathscr{S}}$ $(p \geq 0)$ are acyclic with respect to the functor $\Gamma(X; \bullet)$. Applying Corollary 10.4.2 the result follows. ☐

10.8 Action of groups

G-modules

We will be extremely sketchy on this subject which would deserve a whole book. More details may be found for example in [Bro82] (see also [AW67, Wei94]).

Let G be a group with unit denoted e. We identify G with the category $\mathscr{G}$ with one object, the morphisms of this object being G. We set

$$G\text{-}\mathrm{Mod}(\mathbf{k}) = \mathrm{Fct}(\mathscr{G}, \mathrm{Mod}(\mathbf{k})).$$

Hence, G-$\mathrm{Mod}(\mathbf{k})$ is an abelian category and an object M of this category is a $\mathbf{k}$-module M on which G acts on the left. Note that $\mathrm{Hom}_{G\text{-}\mathrm{Mod}(\mathbf{k})}(M, N)$ is the group of $\mathbf{k}$-linear maps $u\colon M \to N$ which commute to the G-action.

A G-module M is trivial if the action of G is trivial, that is, $g \cdot m = m$ for all $m \in M$ and all $g \in G$. We get an additive functor

$$\mathrm{triv}\colon \mathrm{Mod}(\mathbf{k}) \to G\text{-}\mathrm{Mod}(\mathbf{k}).$$

One has the two functors I^G and I_G from G-$\mathrm{Mod}(\mathbf{k})$ to $\mathrm{Mod}(\mathbf{k})$ defined for $M \in G$-$\mathrm{Mod}(\mathbf{k})$ as follows:

$$\begin{aligned} I^G(M) &= \{m \in M; g\cdot m = m \text{ for all } g \in G\} = \bigcap_{g\in G} \ker(M \xrightarrow{\mathrm{id}-g} M),\\ I_G(M) &= M/\text{module generated by } \{g\cdot m - m; g \in G, m \in M\}. \end{aligned}$$

The module $I^G(M)$ (also denoted M^G in the literature) is thus the submodule of G-invariants of M. Let $N \in \mathrm{Mod}(\mathbf{k})$ and $M \in G$-$\mathrm{Mod}(\mathbf{k})$. One has the isomorphisms, functorial in N and M:

$$\begin{aligned} \mathrm{Hom}_{G\text{-}\mathrm{Mod}(\mathbf{k})}(\mathrm{triv}(N), M) &\simeq \mathrm{Hom}_{\mathbf{k}}(N, I^G(M)),\\ \mathrm{Hom}_{G\text{-}\mathrm{Mod}(\mathbf{k})}(M, \mathrm{triv}(N)) &\simeq \mathrm{Hom}_{\mathbf{k}}(I_G(M), N). \end{aligned}$$

In other words, the functor I^G is right adjoint to the functor triv and I_G is left adjoint to the functor triv. The first isomorphism translates the fact that a $\mathbf{k}$-linear map from N to M which commutes to the trivial action of G on N takes its values in $I^G(M)$. The second isomorphism translates the fact that a $\mathbf{k}$-linear map from M to the module N endowed with the trivial action of G and which commutes with the action of G factorizes through the module generated by $\{g\cdot m - m; g \in G, m \in M\}$. As a particular case of the first isomorphism, we get:

$$I^G(M) \simeq \mathrm{Hom}_{G\text{-}\mathrm{Mod}(\mathbf{k})}(\mathrm{triv}(\mathbf{k}), M). \tag{10.8.1}$$

By the adjunction formulas, we get that triv is exact, I^G is left exact and I_G is right exact.

Denote by $\mathbf{k}G$ the $\mathbf{k}$-group ring associated with the discrete group G, that is, the free $\mathbf{k}$-module with basis G endowed with the algebra structure given by

$$\sum_i a_i g_i \cdot \sum_j b_j h_j = \sum_{i,j} a_i b_j g_i h_j, \quad a_i, b_j \in \mathbf{k}, g_i, h_j \in G \text{ (finite sums)}.$$

There is a natural equivalence of abelian categories

$$\mathrm{Mod}(\mathbf{k}G) \simeq G\text{-}\mathrm{Mod}(\mathbf{k}). \tag{10.8.2}$$

It follows that the category G-$\mathrm{Mod}(\mathbf{k})$ admits enough injectives and the derived functor RI^G is well-defined. One often sets:

$$H^p(G; M) := R^p I^G(M).$$

To calculate it, using (10.8.1) and (10.8.2), one calculates $\mathrm{RHom}_{\mathbf{k}G}(\mathrm{triv}(\mathbf{k}), M)$ using a so-called bar-resolution. This is the free resolution of $\mathrm{triv}(\mathbf{k})$ as a $\mathbf{k}G$-module (see [AW67, p. 96],[Bro82, Ch III, p. 59], [Wei94, § 6.5] for details):

$$\cdots \to \mathbf{k}G^2 \xrightarrow{\delta} \mathbf{k}G \xrightarrow{\varepsilon} \mathbf{k} \to 0$$

where $\varepsilon(g) = 1$ for all $g \in G$.

Assume that G is a cyclic group of finite order n with generator t. Set $N = 1 + t + \cdots + t^{n-1} \in \mathbf{k}G$. We have an exact sequence

$$0 \to \mathbf{k} \xrightarrow{N} \mathbf{k}G \xrightarrow{t-1} \mathbf{k}G \xrightarrow{\varepsilon} \mathbf{k} \to 0.$$

We deduce the 2-periodic long exact sequence

$$\cdots \to \mathbf{k}G \xrightarrow{t-1} \mathbf{k}G \xrightarrow{N} \mathbf{k}G \xrightarrow{t-1} \mathbf{k}G \xrightarrow{\varepsilon} \mathbf{k} \to 0.$$

Example 10.8.1. Let us choose $\mathbf{k} = \mathbb{Z}$, $G = \mathbb{Z}/2\mathbb{Z}$ and $M = \mathbb{Z}$ viewed as a trivial G-module. Using a resolution as above, one can prove:

$$H^p(\mathbb{Z}/2\mathbb{Z};\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p \text{ is even } > 0, \\ 0 \text{ otherwise.} \end{cases}$$

Now denote by $\widetilde{\mathbb{Z}}$ the $\mathbb{Z}$-module $\mathbb{Z}$ endowed with the $\mathbb{Z}/2\mathbb{Z}$-action in which the action of 1mod2 is multiplication by -1. Then

$$H^p(\mathbb{Z}/2\mathbb{Z};\widetilde{\mathbb{Z}}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \text{ if } p \text{ is odd}, \\ 0 \text{ otherwise.} \end{cases}$$

Action on a topological space

Assume now that G is endowed with the discrete topology and acts on a topological space X, that is, we have a continuous map

$$\mu \colon G \times X \to X$$

satisfying $\mu(e,x) = x$, $\mu(g_1 \cdot g_2, x) = \mu(g_2, \mu(g_1, x))$. On X, the relation $x \sim y$ if there exists $g \in G$ such that $\mu(g,x) = y$ is an equivalence relation and one denotes by X/G the quotient space. One sets $Y = X/G$, one endows Y with the quotient topology and one denotes by

$$\rho \colon X \to Y = X/G$$

the quotient map. For $g \in G$, we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "\rho"'] & X \arrow[d, "\rho"] \\ & Y. \end{tikzcd}$$

For $g \in G$, we deduce a morphism of functors (see (10.7.2)):

$$g^\sharp \colon \mathrm{id}_Y \to \rho_* \circ \rho^{-1}, \tag{10.8.3}$$

as the composition

$$\mathrm{id}_Y \to \rho_* \circ g_* \circ g^{-1} \circ \rho^{-1} \simeq \rho_* \circ \rho^{-1}.$$

Applying $\rho_* \circ \rho^{-1}$, we get a morphism

$$g^\sharp \colon \rho_* \circ \rho^{-1} \to \rho_* \circ \rho^{-1}. \tag{10.8.4}$$

Hence, for $F \in \mathrm{Mod}(\mathbf{k}_Y)$, one can consider $\rho_*\rho^{-1}F$ as a G-module. (We do not give a precise meaning to this notion.) In particular, $\Gamma(Y;\rho_*\rho^{-1}F)$ is a G-module.

For $F \in \mathrm{Mod}(\mathbf{k}_Y)$, define $I^G(\rho_*\rho^{-1}F)$ by the exact sequence

$$0 \to I^G(\rho_*\rho^{-1}F) \to \bigcap_{g \in G} \big(\rho_*\rho^{-1}F \xrightarrow{\mathrm{id} - g^\sharp} \rho_*\rho^{-1}F\big).$$

The functor $\Gamma(Y;\bullet)$ being left exact, we get:

Lemma 10.8.2. *Let* $F \in \mathrm{Mod}(\mathbf{k}_Y)$*. Then*

$$I^G(\Gamma(Y;\rho_*\rho^{-1}F)) \simeq \Gamma(Y; I^G(\rho_*\rho^{-1}F)).$$

Since G acts trivially on F, the natural morphism $F \to \rho_*\rho^{-1}F$ factorizes through $I^G(F)$.

Lemma 10.8.3. *Assume that* X *is Hausdorff and* G *is finite and acts freely on* X*. Then, for* $F \in \mathrm{Mod}(\mathbf{k}_Y)$*, one has the isomorphism* $F \xrightarrow{\sim} I^G(\rho_*\rho^{-1}F)$*.*

Proof. Notice first that the fibers of ρ are finite. Let $y \in Y$. The module F_y is isomorphic to the submodule of $\Gamma(\rho^{-1}(y);\rho^{-1}F)$ on which G acts trivially, that is, $F_y \simeq I^G((\rho_*\rho^{-1}F)_y)$. Since $I^G((\rho_*\rho^{-1}F)_y) \simeq (I^G(F))_y$, we get the result. □

Recall that for a space Z, one denotes by a_Z the map $Z \to \mathrm{pt}$. Applying the functor a_{Y*} to (10.8.4), we deduce that there is a well-defined functor

$$a_{X*} \circ \rho^{-1} \colon \mathrm{Mod}(\mathbf{k}_Y) \to G\text{-}\mathrm{Mod}(\mathbf{k}),$$

hence, a functor

$$I^G \circ a_{X*} \circ \rho^{-1} \colon \mathrm{Mod}(\mathbf{k}_Y) \to \mathrm{Mod}(\mathbf{k}).$$

Theorem 10.8.4. *Assume that* X *is Hausdorff and that* G *is finite and acts freely on* X*. Then one has the isomorphism of functors* $RI^G \circ \mathrm{R}a_{X*} \circ \rho^{-1} \simeq \mathrm{R}a_{Y*}$*.*

Proof. (i) The isomorphism $I^G \circ a_{X*} \circ \rho^{-1} \simeq a_{Y*}$ follows from Lemmas 10.8.3 and 10.8.2 .

(ii) It follows from the hypotheses that ρ^{-1} sends injective sheaves to injective sheaves (apply the result of Exercise 9.5), and one knows that a_{X*} sends injective sheaves to injective sheaves. Therefore, the derived functor of the composition is the composition of the derived functors. □

Corollary 10.8.5. *Let* $M \in \mathrm{Mod}(\mathbf{k})$*. Then* $\mathrm{R}\Gamma(Y; M_Y) \simeq RI^G\mathrm{R}\Gamma(X; M_X)$*.*

10.9 Cohomology of some classical manifolds

Here, $\mathbf{k}$ denotes as usual a commutative unitary ring and M denotes a $\mathbf{k}$-module.

1-sphere

Recall Example 8.8.4: X is the circle $\mathbb{S}^1$, U_1 and U_2 are two open intervals covering $\mathbb{S}^1$. Then $U_1 \cap U_2$ has two connected components U_{12}^+ and U_{12}^-. Moreover $\mathbf{k}$ is a field, $\alpha \in \mathbf{k}^\times$ and L_α denotes the locally constant sheaf of rank one over $\mathbf{k}$ obtained by glueing $\mathbf{k}_{U_1}$ and $\mathbf{k}_{U_2}$ by the identity on U_{12}^+ and by multiplication by $\alpha \in \mathbf{k}^\times$ on U_{12}^-.

Let $\{Z_j\}_{j=0,1,2}$ be a closed covering by 3 intervals such that the Z_{ij}'s are single points and $Z_{012} = \varnothing$. Applying Theorem 10.7.9, we find that if F is a locally constant sheaf on X, the cohomology groups $H^j(X;F)$ are the cohomology objects of the complex:

$$0 \to F_{Z_0} \oplus F_{Z_1} \oplus F_{Z_2} \xrightarrow{d} F_{Z_{12}} \oplus F_{Z_{20}} \oplus F_{Z_{01}} \to 0.$$

Then for $j = 0$ (resp. for $j = 1$), $H^j(\mathbb{S}^1; L_\alpha)$ is the kernel (resp. the cokernel) of the matrix $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -\alpha \\ -1 & 1 & 0 \end{pmatrix}$ acting on $\mathbf{k}^3$. (See Example 10.4.3.) Note that these kernel and cokernel are zero except in case of $\alpha = 1$ which corresponds to the constant sheaf $\mathbf{k}_X$.

n-sphere

Consider the topological n-sphere $\mathbb{S}^n$. Recall that it can be defined as follows. Let $\mathbb{E}$ be an $\mathbb{R}$-vector space of dimension $n+1$ and denote by $\dot{\mathbb{E}}$ the set $\mathbb{E} \setminus \{0\}$. Then

$$\mathbb{S}^n \simeq \dot{\mathbb{E}}/\mathbb{R}^+,$$

where $\mathbb{R}^+$ denotes the multiplicative group of positive real numbers and $\mathbb{S}^n$ is endowed with the quotient topology. In other words, $\mathbb{S}^n$ is the set of all half-lines in $\mathbb{E}$. If one chooses an Euclidian norm on $\mathbb{E}$, then one may identify $\mathbb{S}^n$ with the unit sphere in $\mathbb{E}$.

We have $\mathbb{S}^n = \bar{D}^+ \cup \bar{D}^-$, where $\bar{D}^+$ and $\bar{D}^-$ denote the closed hemispheres, and $\bar{D}^+ \cap \bar{D}^- \simeq \mathbb{S}^{n-1}$. Let us prove that:

$$\mathrm{R}\Gamma(\mathbb{S}^n; \mathbf{k}_{\mathbb{S}^n}) = \mathbf{k} \oplus \mathbf{k}[-n]. \tag{10.9.1}$$

Consider the Mayer-Vietoris long exact sequence

$$\begin{aligned} \to H^j(\bar{D}^+; \mathbf{k}_{\bar{D}^+}) \oplus H^j(\bar{D}^-; \mathbf{k}_{\bar{D}^-}) &\to H^j(\mathbb{S}^{n-1}; \mathbf{k}_{\mathbb{S}^{n-1}}) \\ &\to H^{j+1}(\mathbb{S}^n; \mathbf{k}_{\mathbb{S}^n}) \to \cdots \end{aligned} \tag{10.9.2}$$

The closed hemispheres being contractible, their cohomology is concentrated in degree 0. Then we find by induction on n that the cohomology of $\mathbb{S}^n$ is concentrated in degree 0 and n and isomorphic to $\mathbf{k}$ in these degrees. To conclude that $\mathrm{R}\Gamma(\mathbb{S}^n; \mathbf{k}_{\mathbb{S}^n})$ is the direct sum of its cohomology objects, use the fact that $\mathrm{Ext}^j_{\mathrm{D}^b(\mathbf{k})}(\mathbf{k}, \mathbf{k}) \simeq 0$ for $j \neq 0$ and Exercise 7.6.

Let $\mathbb{E}$ be a real vector space of dimension $n+1$, and let $X = \mathbb{E} \setminus \{0\}$. Assume $\mathbb{E}$ is endowed with a norm $|\cdot|$. The map $x \mapsto x((1-t) + t/|x|)$ defines an homotopy of X with the sphere $\mathbb{S}^n$. Hence the cohomology of a constant sheaf with stalk M on $V \setminus \{0\}$ is the same as the cohomology of the sheaf $M_{\mathbb{S}^n}$

As an application, one obtains that the dimension of a finite dimensional vector space is a topological invariant. In other words, if V and W are two real finite dimensional vector spaces and are topologically isomorphic, they have the same dimension. In fact, if V has dimension n, then $V \setminus \{0\}$ is homotopic to $\mathbb{S}^{n-1}$.

Notice that $\mathbb{S}^n$ is not contractible, although one can prove that any locally constant sheaf on $\mathbb{S}^n$ for $n \geq 2$ is constant.

Denote by a the antipodal map on $\mathbb{S}^n$ (the map deduced from $x \mapsto -x$) and denote by $a^{\sharp n}$ the action of a on $H^n(\mathbb{S}^n; M_{\mathbb{S}^n})$. Using (10.9.2), one deduces the commutative diagram:

$$\begin{tikzcd} H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \arrow[r, "u"] \arrow[d, "a^{\sharp n-1}"'] & H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \arrow[d, "a^{\sharp n}"'] \\ H^{n-1}(\mathbb{S}^{n-1}; M_{\mathbb{S}^{n-1}}) \arrow[r, "-u"] & H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \end{tikzcd} \tag{10.9.3}$$

For $n = 1$, the map a is homotopic to the identity (in fact, it is the same as a rotation of angle π). By (10.9.3), we deduce:

$$a^{\sharp n} \text{ acting on } H^n(\mathbb{S}^n; M_{\mathbb{S}^n}) \text{ is } (-)^{n+1}. \tag{10.9.4}$$

n-torus

The Künneth formula will be proved in the next chapter (see Corollary 11.3.8). It allows us to calculate the cohomology of the n-torus $\mathbb{T}^n := (\mathbb{S}^1)^n$. Applying (10.9.1) for $n = 1$ we get:

$$\mathrm{R}\Gamma(\mathbb{T}^n; \mathbf{k}_{\mathbb{T}^n}) \simeq (\mathbf{k} \oplus \mathbf{k}[-1])^{\otimes n}. \tag{10.9.5}$$

For example, for $n = 2$, we find

$$\begin{aligned}(\mathbf{k} \oplus \mathbf{k}[-1]) \otimes (\mathbf{k} \oplus \mathbf{k}[-1]) &\simeq \big(\mathbf{k} \otimes (\mathbf{k} \oplus \mathbf{k}[-1])\big) \oplus \big(\mathbf{k}[-1] \otimes (\mathbf{k} \oplus \mathbf{k}[-1])\big) \\ &\simeq \big(\mathbf{k} \oplus \mathbf{k}[-1])\big) \oplus \big(\mathbf{k}[-1] \oplus \mathbf{k}[-2]\big) \\ &\simeq \mathbf{k} \oplus \mathbf{k}^{\oplus 2}[-1] \oplus \mathbf{k}[-2].\end{aligned}$$

Real projective spaces I [1]

Let $\mathbb{P}^n$ denote the real n-dimensional projective space, that is $\mathbb{P}^n = \mathbb{R}^{n+1}/\mathbb{R}^\times$. This space $\mathbb{P}^n$ may also be obtained as the quotient of the sphere $\mathbb{S}^n$ by the antipodal map $a : x \mapsto -x$, that is, the quotient of $\mathbb{S}^n$ by the group $\mathbb{Z}/2\mathbb{Z}$. Hence $\mathbb{S}^n$ is a twofold covering of $\mathbb{P}^n$. Hence, we may apply Theorem 10.8.4 with $X = \mathbb{S}^n$, $Y = \mathbb{P}^n$, $\rho\colon X \to Y$ and $G = \mathbb{Z}/2\mathbb{Z}$. We choose $\mathbf{k} = \mathbb{Z}$.

We have a distinguished triangle in the derived category of $\mathbb{Z}/2\mathbb{Z}$-modules:

$$\tau^{<n}\mathrm{R}a_{X*}\mathbb{Z}_X \to \mathrm{R}a_{X*}\mathbb{Z}_X \to H^n(\mathrm{R}a_{X*}\mathbb{Z}_X)\,[-n] \xrightarrow{+1} .$$

Since $X = \mathbb{S}^n$ and $\mathbb{Z}_X \simeq \rho^{-1}\mathbb{Z}_Y$, this triangle reduces to

$$\mathbb{Z} \to \mathrm{R}a_{X*}\rho^{-1}\mathbb{Z}_Y \to \widetilde{\mathbb{Z}}\,[-n] \xrightarrow{+1}$$

where $\widetilde{\mathbb{Z}} = \mathbb{Z}$, the action of $\mathbb{Z}/2\mathbb{Z}$ is trivial on $\mathbb{Z}$, is trivial on $\widetilde{\mathbb{Z}}$ if n is odd and this action on $\widetilde{\mathbb{Z}}$ is the multiplication by -1 if n is even (we apply (10.9.4)).

Now we apply the functor RI^G to this triangle. We get the d. t.

$$RI^G(\mathbb{Z}) \to \mathrm{R}\Gamma(Y; \mathbb{Z}_Y) \to RI^G(\widetilde{\mathbb{Z}})\,[-n] \xrightarrow{+1} .$$

Using the fact that the cohomology of $\mathbb{Z}_Y$ is concentrated in degree $[0, n]$ (see Proposition 11.5.2 below), we get

$$H^p(Y; \mathbb{Z}_Y) \simeq R^pI^G(\mathbb{Z}) \text{ for } p < n$$

and an exact sequence

$$0 \to R^nI^G(\mathbb{Z}) \to H^n(Y; \mathbb{Z}_Y) \to R^0I^G(\widetilde{\mathbb{Z}}) \to R^{n+1}I^G(\mathbb{Z}) \to 0.$$

---

[1] The classical proofs calculating the cohomology of the real projective space uses spectral sequences. The proof proposed here (including some results of § 10.8), using truncation functors, is much shorter. It is due to Tony Yue Yu who did it when he was a Master 2 student at UPMC around 2011.

Using the results of Example 10.8.1, we find:

If n is odd, $R^nI^G(\mathbb{Z}) \simeq 0$, $R^{n+1}I^G(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$ and $R^0I^G(\mathbb{Z}) \simeq \mathbb{Z}$.

Therefore, we have an exact sequence $0 \to H^n(Y;\mathbb{Z}_Y) \to \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$ which implies $H^n(Y;\mathbb{Z}_Y) \simeq \mathbb{Z}$.

If n is even, $R^nI^G(\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$, $R^{n+1}I^G(\mathbb{Z}) \simeq 0$ and $R^0I^G(\widetilde{\mathbb{Z}}) \simeq 0$.

To sumarize, we find:

$$\text{(10.9.6)} \qquad \text{if } n \text{ is odd: } H^p(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, n, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p = 2, 4, \dots, n-1, \\ 0 \text{ otherwise,} \end{cases}$$

$$\text{(10.9.7)} \qquad \text{if } n \text{ is even: } H^p(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} \text{ if } p = 0, \\ \mathbb{Z}/2\mathbb{Z} \text{ if } p = 2, 4, \dots, n, \\ 0 \text{ otherwise.} \end{cases}$$

Real projective spaces II

Let us give another, and more elementary, proof of the preceding result. We have a sequence of morphisms $\mathrm{R}\Gamma(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \to \mathrm{R}\Gamma(\mathbb{S}^n;\mathbb{Z}_{\mathbb{S}^n}) \to \mathrm{R}\Gamma(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n})$ and the composition is multiplication by 2. If one chooses a hyperplane in $\mathbb{R}^{n+1}$, its image in $\mathbb{S}^n$ is a sphere $\mathbb{S}^{n-1}$ and its image in $\mathbb{P}^n$ is a projective space $\mathbb{P}^{n-1}$. We get a decomposition $\mathbb{S}^n = D^+ \sqcup D^- \sqcup \mathbb{S}^{n-1}$, $\mathbb{P}^n = D \sqcup \mathbb{P}^{n-1}$ where D^+, D^- and D are all isomorphic to $\mathbb{R}^n$. Hence we get the commutative diagram of distinguished triangles:

$$\text{(10.9.8)} \quad \begin{tikzcd} \mathrm{R}\Gamma_c(D;\mathbb{Z}_D) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathbb{P}^{n-1};\mathbb{Z}_{\mathbb{P}^{n-1}}) \arrow[r, "+1"] \arrow[d] & {} \\ \mathrm{R}\Gamma_c(D^+ \sqcup D^-;\mathbb{Z}_{D^+\sqcup D^-}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathbb{S}^n;\mathbb{Z}_{\mathbb{S}^n}) \arrow[r] \arrow[d] & \mathrm{R}\Gamma(\mathbb{S}^{n-1};\mathbb{Z}_{\mathbb{S}^{n-1}}) \arrow[r, "+1"] \arrow[d] & {} \\ \mathrm{R}\Gamma_c(D;\mathbb{Z}_D) \arrow[r] & \mathrm{R}\Gamma(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \arrow[r] & \mathrm{R}\Gamma(\mathbb{P}^{n-1};\mathbb{Z}_{\mathbb{P}^{n-1}}) \arrow[r, "+1"] & {} \end{tikzcd}$$

and the composition of two vertical arrows is multiplication by 2. Let us recover (10.9.6) and (10.9.7).

Since $\mathbb{P}^0 = \{pt\}$, and $\mathbb{P}^1 \simeq \mathbb{S}^1$, the result is clear for $n = 0, 1$. Arguing by induction, we shall prove:

$$(a_n)\colon \qquad H^n(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \simeq \begin{cases} \mathbb{Z} & \text{if } n \text{ is odd} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is even} \end{cases}$$

$$(b_n)\colon \qquad H^{n-1}(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \simeq \begin{cases} 0 & \text{if } n \text{ is even} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n \text{ is odd} \end{cases}$$

$(c_n)\colon$ $H^n(\mathbb{P}^n;\mathbb{Z}_{\mathbb{P}^n}) \to H^n(\mathbb{S}^n;\mathbb{Z}_{\mathbb{S}^n})$ is multiplication by 2.

Assume $(a_{n-1}), (b_{n-1}), (c_{n-1})$. The morphisms of triangles in (10.9.8) give rise to the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^{n-1}(\mathbb{P}^n; \mathbb{Z}_{\mathbb{P}^n}) \arrow[r] \arrow[d] & H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}_{\mathbb{P}^{n-1}}) \arrow[r, "\lambda"] \arrow[d] & \mathbb{Z} \arrow[r, "\mu"] \arrow[d, "\delta"'] & H^n(\mathbb{P}^n; \mathbb{Z}_{\mathbb{P}^n}) \arrow[r] \arrow[d, "\varepsilon"'] & 0 \\
 & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "\alpha"] \arrow[d] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\beta"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\varepsilon'"'] & 0 \\
0 \arrow[r] & H^{n-1}(\mathbb{P}^n; \mathbb{Z}_{\mathbb{P}^n}) \arrow[r] & H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}_{\mathbb{P}^{n-1}}) \arrow[r, "\lambda"] & \mathbb{Z} \arrow[r, "\mu"] & H^n(\mathbb{P}^n; \mathbb{Z}_{\mathbb{P}^n}) \arrow[r] & 0
\end{tikzcd}$$

(i) First, assume n is odd. Then $H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}_{\mathbb{P}^{n-1}}) \simeq \mathbb{Z}/2\mathbb{Z}$ and λ is the zero morphism, μ the identity (up to sign). This proves (a_n) and (b_n). Since ε is not zero (the composition $\varepsilon' \circ \varepsilon$ is multiplication by 2 in $\mathbb{Z}$), the composition $\beta \circ \delta$ is not zero, and since $\beta(y, z) = y - z$ and $\delta(x) = (x, x)$ or $(x, -x)$ (up to sign), we find $\beta \circ \delta = 2$, hence $\varepsilon = 2$.

(ii) Now assume n is even. Then $H^{n-1}(\mathbb{P}^{n-1}; \mathbb{Z}_{\mathbb{P}^{n-1}}) \simeq \mathbb{Z}$ and since λ is not zero ($\delta \circ \lambda \neq 0$), we get (b_n). Since $\delta(x) = (x, x)$ or$(x, -x)$ (up to sign), we get $\lambda(x) = 2x$, hence (a_n), and (c_n).

Exercises to Chapter 10

Exercise 10.1. Let X be a topological space. Prove that the natural morphism $(\mathscr{H}om\,(F, G))_x \to \mathrm{Hom}\,(F_x, G_x)$ is not an isomorphism in general. (Hint: choose $F = \mathbf{k}_{XU}$ with U open.)

Exercise 10.2. Assume that $X = \mathbb{R}$, let S be a closed interval and let $U = X \setminus S$. Assume both S and U non-empty.

(i) Prove that the natural map $\Gamma(X; \mathbf{k}_X) \to \Gamma(X; \mathbf{k}_{XS})$ is surjective and deduce that $\Gamma(X; \mathbf{k}_{XU}) \simeq 0$.

(ii) Prove that the morphism $\mathbf{k}_X \to \mathbf{k}_{XS}$ does not split.

Exercise 10.3. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Define $\widetilde{F} \in \mathrm{Mod}(\mathbf{k}_X)$ by $\widetilde{F} = \bigoplus_{x \in X} F_{\{x\}}$. (Here, $F_{\{x\}} \in \mathrm{Mod}(\mathbf{k}_X)$ and the direct sum is calculated in $\mathrm{Mod}(\mathbf{k}_X)$, not in $\mathrm{PSh}(\mathbf{k}_X)$.) Prove that F_x and $\widetilde{F}_x$ are isomorphic for all $x \in X$, although F and $\widetilde{F}$ are not isomorphic in general.

Exercise 10.4. Let $Z = Z_1 \sqcup Z_2$ be the disjoint union of two sets Z_1 and Z_2 in X.

(i) Assume that Z_1 and Z_2 are both open (resp. closed) in X. Prove that $\mathbf{k}_{XZ} \simeq \mathbf{k}_{XZ_1} \oplus \mathbf{k}_{XZ_2}$.

(ii) Give an example which shows that (i) is no more true if one only assume that Z_1 and Z_2 are both locally closed.

Exercise 10.5. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$, $S = \{(x, y) \in X; xy \geq 1\}$, and let $f \colon X \to Y$ be the map $(x, y) \mapsto y$. calculate $f_* \mathbf{k}_{XS}$.

Exercise 10.6. Let X be a topological space, $\{x, \dots, x_m\}$ m distinct points. Calculate $H^j(X \setminus \{x, \dots, x_m\}; \mathbf{k}_X)$ when $X = \mathbb{R}^n$ and when $X = \mathbb{S}^n$, $n > 1$.

Exercise 10.7. Let X be a topological space, M a closed subspace and F a sheaf on X. Assume there is an $n > 0$ such that $H^j_M(F) \simeq 0$ for $j < n$. Prove that the presheaf $U \mapsto H^n_{M\cap U}(U;F)$ is a sheaf and is isomorphic to the sheaf $H^n_M(F)$. (See Notation 10.2.12.)

Exercise 10.8. Let X be a locally compact space, M a closed subspace and F a sheaf on X. Assume there is an $n > 0$ such that for any compact $K \subset M$, $H^j_K(X;F) = 0$ for all $j < n$ and that for each pair $K_1 \subset K_2$ of compact subsets of M, the natural morphism $H^n_{K_1}(X;F) \to H^n_{K_2}(X;F)$ is injective.

(i) Prove that for each relatively compact open subset ω of M, $H^j_\omega(X;F) = 0$ for all $j < n$, and the presheaf $\omega \mapsto H^n_\omega(X;F)$ is the sheaf $H^n_M(F)$.

(ii) Prove that if $K \subset M$ is compact, $\Gamma_K(M; H^n_M(F)) \simeq H^n_K(X;F)$.

(iii) Assume moreover that $H^j_K(X;F) = 0$ for all compact subsets of M and all $j > n$. Prove that the sheaf $H^n_M(F)$ is flabby.

Remark: when M is a real analytic manifold of dimension n, X a complexification, and $F = \mathscr{O}_X$, all hypotheses are satisfied. The sheaf $H^n_M(\mathscr{O}_X) \otimes or_M$ is called the sheaf of Sato's hyperfunctions. Here or_M is the orientation sheaf, see § 11.5.

Exercise 10.9. In this exercise, we shall admit the following theorem: for any open subset U of the complex line $\mathbb{C}$, one has $H^j(U;\mathscr{O}_\mathbb{C}) \simeq 0$ for $j > 0$.

Let ω be an open subset of $\mathbb{R}$, and let $U_1 \subset U_2$ be two open subsets of $\mathbb{C}$ containing ω as a closed subset.

(i) Prove that the natural map $\mathscr{O}(U_2 \setminus \omega)/\mathscr{O}(U_2) \to \mathscr{O}(U_1 \setminus \omega)/\mathscr{O}(U_1)$ is an isomorphism. One denote by $\mathscr{B}(\omega)$ this quotient.

(ii) Construct the restriction morphism to get the presheaf $\omega \to \mathscr{B}(\omega)$, and prove that this presheaf is a sheaf (the sheaf $\mathscr{B}_\mathbb{R}$ of Sato's hyperfunctions on $\mathbb{R}$).

(iii) Prove that the restriction morphisms $\mathscr{B}(\mathbb{R}) \to \mathscr{B}(\omega)$ are surjective (i.e. the sheaf $\mathscr{B}_\mathbb{R}$ is flabby).

(iv) Let Ω an open subset of $\mathbb{C}$ and let $P = \sum_{j=1}^m a_j(z)\frac{\partial^j}{\partial^j z}$ be a holomorphic differential operator (the coefficients are holomorphic in Ω). Recall the Cauchy theorem which asserts that if Ω is simply connected and if $a_m(z)$ does not vanish on Ω, then P acting on $\mathscr{O}(\Omega)$ is surjective. Prove that if ω is an open subset of $\mathbb{R}$ and if P is a holomorphic differential operator defined in a open neighborhood of ω, then P acting on $\mathscr{B}(\omega)$ is surjective

Exercise 10.10. Let M be a $\mathbf{k}$-module and let X be an open subset of $\mathbb{R}^n$. Let F be a presheaf such that for any non empty convex open subsets $U \subset X$, there exists an isomorphism $F(U) \simeq M$ and this isomorphism is compatible to the restriction morphisms for $V \subset U$. Prove that the associated sheaf is locally constant.

Exercise 10.11. We shall recover here Proposition 5.2.6.

Let $X = \mathbb{N}$ endowed with the topology for which the open subsets are the intervals $[0,\dots,n], n \geq -1$ and $\mathbb{N}$.

(i) Prove that a presheaf F of $\mathbf{k}$-modules on X is nothing but a projective system $(F_n, \rho_{m,n})$ indexed by $\mathbb{N}$ and that this presheaf is a sheaf if and only if $F(X) \simeq \varprojlim_n F_n$.

(ii) Prove that if $F_{n+1} \to F_n$ is onto, then the sheaf F is flabby.

(iii) Deduce that if $0 \to M'_n \to M_n \to M''_n \to 0$ is an exact sequence of projective systems of $\mathbf{k}$-modules and the morphisms $M'_{n+1} \to M'_n$ are onto, then the sequence $0 \to \lim\limits_n M'_n \to \lim\limits_n M_n \to \lim\limits_n M''_n \to 0$ is exact.

(iv) Prove that for any sheaf F on X there exists an exact sequence $0 \to F \to F_0 \to F_1 \to 0$, with F_0 and F_1 flabby.

(v) By considering the ordered set $\mathbb{N}$ as a category, denote by π the left exact functor

$$\pi\colon (\mathrm{Fct}(\mathbb{N}^{\mathrm{op}}, \mathrm{Mod}(\mathbf{k})) \to \mathrm{Mod}(\mathbf{k}), \quad \alpha \mapsto \lim \alpha. \tag{10.9.9}$$

Prove that $R^j\pi \simeq 0$ for $j > 1$.

Exercise 10.12. let X be a real n-dimensional vector space and let U be an open convex subset, $j\colon U \hookrightarrow X$ the embedding. Calculate $\mathrm{R}j_*\mathbf{k}_U$.

Exercise 10.13. Assume that $\mathbf{k}$ is a field.

(i) Let F be a subsheaf of the constant sheaf $\mathbf{k}_X$. Prove that there exists an open subset U of X such that $F \simeq \mathbf{k}_{XU}$.

(ii) Deduce from (i) that if U is open and $\varphi\colon \mathbf{k}_U \to G$ is a morphism, then the image of $\mathbf{k}_U$ is isomorphic to $\mathbf{k}_{XZ}$ for a locally closed subset Z of X.

(iii) Prove that a sheaf of $\mathbf{k}_X$-modules is injective if and only if it is flabby. (Hint: use (i) and Exercise 9.5.)

Exercise 10.14. By considering the space $X = \mathbb{S}^1$ and the map $a_X\colon X \to \mathrm{pt}$, prove that the morphism $R(f_* \circ f^{-1}) \to \mathrm{R}f_* \circ f^{-1}$ is not an isomorphism in general.

Exercise 10.15. Let K be a compact relatively Hausdorff subset of a topological space X. Prove the isomorphism $H^j(K; F) \simeq \operatorname*{colim}\limits_U H^j(U; F)$, where U ranges through the family of open neighborhoods of K.

Exercise 10.16. By using (10.9.1), calculate $\mathrm{R}\Gamma(X; \mathbf{k}_X)$ for $X = \mathbb{S}^n \times \mathbb{S}^n$.

Exercise 10.17. Assume $\mathbf{k}$ is a field, and for $\alpha \in \mathbf{k}^\times$ let L_α be the locally free sheaf of rank one on $\mathbb{S}^1$ constructed in Example 8.8.4. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Calculate $\mathrm{R}\Gamma(X; L_\alpha \boxtimes L_\beta)$ for $\alpha, \beta \in \mathbf{k}^\times$.

Exercise 10.18. Let $\bar{D}$ denote the closed disc in $\mathbb{R}^2$ with boundary $\mathbb{S}^1$. Let $\iota : \mathbb{S}^1 \hookrightarrow \bar{D}$ denote the embedding. Prove that there exists no continuous map $f : \bar{D} \to \mathbb{S}^1$ such that the composition $f \circ \iota$ is the identity.

Exercise 10.19. Let Y and Y' be two topological spaces, S and S' two closed subsets of Y and Y' respectively, $f : S \simeq S'$ a topological isomorphism. Define the topological space $X := Y \sqcup_S Y'$ as the quotient $Y \sqcup Y'/\sim$ where $\sim$ is the equivalence relation which identifies $x \in Y$ and $y \in Y'$ for $x \in S$, $y \in S'$ and $f(x) = y$.

Let $\mathbb{S}^n$ be the unit sphere of the Euclidian space $\mathbb{R}^{n+1}$, Z the intersection of $\mathbb{S}^n$ with an open ball of radius ε ($0 < \varepsilon << 1$) centered in some point of $\mathbb{S}^n$ and let Σ denote its boundary in $\mathbb{S}^n$. Set $Y = \mathbb{S}^n \setminus Z$, $S = \Sigma$ denote by Y' and S' another copy of Y and S.

(i) Calculate $\mathrm{R}\Gamma(Y \sqcup_S Y'; \mathbf{k}_{Y \sqcup_S Y'})$.

(ii) Same question when replacing the sphere $\mathbb{S}^n$ by the torus $\mathbb{T}^2$ embedded in $\mathbb{R}^3$.

Chapter 11

Duality on locally compact spaces

This chapter is devoted to duality on locally compact spaces. We first define the proper direct image functor $f_!$ associated with a morphism $f\colon X \to Y$ of locally compact spaces. (The definition that we propose here, although equivalent, is not the traditional one.) Next, we prove that c-soft sheaves are acyclic for the functor $f_!$ and we study its derived functor $\mathrm{R}f_!$. We prove the two main results of this theory, namely the projection formula and the base change formula. As a byproduct, we get the Künneth formula.

The existence of the right adjoint $f^!$ to $\mathrm{R}f_!$ follows from the Brown representability theorem. We study the properties of this functor and introduce in particular the dualizing complex ω_X that we explicitly calculate when X is a topological manifold. We also describe the De Rham cohomology on real manifolds, the Dolbeault-Grothendieck cohomology on complex manifolds and we construct the Leray-Grothendieck residues morphism.

Recall that we assume that $\mathbf{k}$ has finite global dimension.

11.1 Proper direct images

In this chapter, unless otherwise stated, all sites X, Y, etc. are locally compact topological spaces.

Definition 11.1.1. Let X and Y be two topological spaces. A continuous map $f\colon X \to Y$ is proper if f is closed (*i.e.*, the image of any closed subset in X is closed in Y) and its fibers are relatively Hausdorff and compact.

If X and Y are locally compact, f is proper if and only if the inverse image of a compact subset of Y is compact in X. If $Y = \mathrm{pt}$, f is proper if and only if X is compact.

Proposition 11.1.2. *Assume that $f\colon X \to Y$ is proper. Then the morphism*

$$(f_*F)_y \to \Gamma(f^{-1}(y); F|_{f^{-1}(y)}). \tag{11.1.1}$$

is an isomorphism.

Proof. When V ranges over the family of open neighborhoods of y, $f^{-1}(V)$ ranges over a neighborhood system of $f^{-1}(y)$. Applying Proposition 10.1.2 we get:

$$\begin{aligned}(f_*F)_y &\simeq \operatorname*{colim}_{V\ni y} \Gamma(f^{-1}V; F)\\ &\xrightarrow{\sim} \Gamma(f^{-1}(y); F|_{f^{-1}(y)}).\end{aligned}$$

□

One says that a map f is finite if it is proper and moreover the inverse image of a finite set is finite.

Corollary 11.1.3. *If f is finite, the functor f_* is exact.*

Lemma 11.1.4. *Let K be a relatively Hausdorff compact subset of X and let $\{F_i\}_{i\in I}$ be a small filtered inductive system of sheaves on X. Then the natural morphism $u\colon \operatorname{colim}_i \Gamma(K;F_i) \to \Gamma(K;\operatorname{colim}_i F_i)$ is an isomorphism.*

Proof. (i) u is a monomorphism. Let $s \in \operatorname{colim}_i \Gamma(K;F_i)$. We may represent s by a section (still denoted s) of $\Gamma(K;F_j)$ for some $j \in I$. Assume that $u(s) = 0$. Then for each $x \in K$, the germ $u(s)_x$ is 0 in $(\operatorname{colim}_i F_i)_x \simeq \operatorname{colim}_i (F_i)_x$. Hence, for each $x \in K$ there exists some $i_x \in I$ and a morphism $j \to i_x$ such that the image of the germ s_x is 0 in F_{i_x}. It follows that the image of s in $\Gamma(U_x;F_{i_x})$ is 0 for an open neighborhood U_x of x. Let us choose a finite covering $\{U_a\}_{a\in A}$ extracted for the covering $\{U_x\}_{x\in K}$. The category I being filtered, there exists $k \in I$ and morphisms $i_x \to k$ such that the image of s in $\Gamma(U_a;F_k)$ is 0 for all $a \in A$. Therefore, $s = 0$.

(ii) u is an epimorphism. Denote by $\widetilde{F}$ the presheaf $U \mapsto \operatorname{colim}_i F_i(U)$ and by F the sheaf $\operatorname{colim}_i F_i$. We have to check that $\Gamma(K;\widetilde{F}) \to \Gamma(K;F)$ is an epimorphism. Since $F \simeq \widetilde{F}^a$, for each $s \in \Gamma(K;F)$ there exists a covering $K = \bigcup_{a\in A} U_a$ and sections $s_a \in \Gamma(U_a;\widetilde{F})$ such that $s|_{U_a} = s_a$. We may assume A is finite and there exists $i \in I$ such that $s_a \in \Gamma(U_a;F_i)$. These sections define $s_i \in \Gamma(K;F_i)$ whose image in $\Gamma(K;F)$ is s. □

Lemma 11.1.5. *Let $f\colon X \to Y$ be a morphism of locally compact spaces and let $\{F_i\}_{i\in I}$ be a small filtered inductive system of sheaves on X. Let $Z \subset X$ be a closed subset and assume that the map $f|_Z$ is proper. Then the natural morphism $\operatorname{colim}_i f_*(F_i)_Z \to f_*(\operatorname{colim}_i (F_i)_Z)$ is an isomorphism.*

Proof. We shall apply Lemma 11.1.4. Let K be a compact subset of Y. One has

$$\begin{aligned}\Gamma(K;f_*(\operatorname{colim}_i (F_i)_Z)) &\simeq \Gamma(f^{-1}K\cap Z;\operatorname{colim}_i (F_i)_Z)\\ &\simeq \operatorname{colim}_i \Gamma(f^{-1}K\cap Z;(F_i)_Z)\\ &\simeq \operatorname{colim}_i \Gamma(K;f_*((F_i)_Z)).\end{aligned}$$

By choosing for K a fundamental neighborhood system of $y \in Y$ we get that the natural morphism of the statement induces an isomorphism on the talks at each $y \in Y$. To conclude, apply Corollary 9.1.9. □

Definition 11.1.6. Let $f\colon X \to Y$ be a morphism of locally compact spaces and let $F \in \operatorname{Mod}(\mathbf{k}_X)$.

(a) One defines the functor $f_!\colon \operatorname{Mod}(\mathbf{k}_X) \to \operatorname{Mod}(\mathbf{k}_Y)$ by setting for $F \in \operatorname{Mod}(\mathbf{k}_X)$:

$$f_!F = \operatorname*{colim}_{U\subset\subset X} f_*(F_U)$$

where U ranges over the family of relatively compact open subsets of X.

(b) One sets $\Gamma_c(X;\,\bullet\,) = a_{X!}(\,\bullet\,)$, where $a_X\colon X\to \mathrm{pt}$.

Proposition 11.1.7. (i) *In the situation of Definition 11.1.6, one has for V an open subset of Y:*

$$\Gamma(V;f_!F)\simeq \operatorname*{colim}_{Z}\Gamma_Z(f^{-1}(V);F)$$

where Z ranges through the family of closed subsets of $f^{-1}(V)$ such that $f|_Z\colon Z\to V$ is proper. In particular, $\Gamma_c(X;F)\simeq \operatorname*{colim}_{K}\Gamma_K(X;F)$, where K ranges through the family of compact subsets of X.

(ii) *If f is proper on $\mathrm{supp}(F)$, then $f_!F\xrightarrow{\sim} f_*F$. In particular, if f is proper, then $f_!\xrightarrow{\sim} f_*$.*

(iii) *The functor $f_!$ is left exact and commutes with small filtered inductive limits.*

(iv) *Let $g\colon Y\to Z$ be a continuous map of locally compact spaces. Then $f_!\circ g_!\simeq (f\circ g)_!$.*

(v) *Let $i_U\colon U\hookrightarrow X$ be an open embedding. Then the functor $i_{U!}$, as given by Definition 11.1.6, is equal to the functor j_U^{-1} (see Notation 8.7.9).*

Proof. (i) We shall apply Lemma 11.1.4. For any relatively compact open subset W of V

$$\begin{aligned}\Gamma(\overline{W};\operatorname*{colim}_{U}f_*F_U)&\simeq \operatorname*{colim}_{U}\Gamma(\overline{W};f_*F_U)\\ &\simeq \operatorname*{colim}_{U,W'}\Gamma(f^{-1}(W');F_U)\end{aligned}$$

where U ranges over the family of relatively compact open subsets of X and W' over the family of open neighbourhoods of $\overline{W}$.

Let $s\in\Gamma(V;f_*F)$.Then $s\in\Gamma(V;f_!F)$ if and only if for any $W\subset V$ open and relatively compact in V, there exists $U\subset X$ open and relatively compact such that $\mathrm{supp}(s)\cap f^{-1}W$ is contained U. This is equivalent to saying that the support of s is proper over Y.

(ii) is obvious.

(iii) The functor $F\mapsto F_U$ is exact, the functor f_* is left exact and the functor colim over small filtered categories is exact. Hence, $f_!$ is left exact. It commutes with small filtered inductive limits by Lemma 11.1.4.

(iv) In the sequel, U ranges over the family of relatively compact open subsets of X, and similarly with V in Y.

By Proposition 10.2.1), the functor $F\mapsto F_U$ commutes with inductive limits and by Lemma 11.1.4 the functor $g_*((\,\bullet\,)_V)$ commutes with filtered inductive limits. Therefore:

$$\begin{aligned}g_!f_!F&\simeq \operatorname*{colim}_{V}g_*((\operatorname*{colim}_{U}f_*F_U)_V)\simeq \operatorname*{colim}_{V}g_*(\operatorname*{colim}_{U}(f_*F_U)_V)\\ &\simeq \operatorname*{colim}_{V}(\operatorname*{colim}_{U}g_*(f_*F_U)_V)\simeq \operatorname*{colim}_{U}g_*f_*F_U\simeq (g\circ f)_*F.\end{aligned}$$

(v) For $F\in\mathrm{Mod}(\mathbf{k}_U)$, we have $\operatorname*{colim}_{V}F_V\xrightarrow{\sim}F$, where V ranges over the family of relatively compact open subsets of U. Hence,

$$j_U^{-1}F\simeq j_U^{-1}\operatorname*{colim}_{V}F_V\simeq \operatorname*{colim}_{V}j_U^{-1}F_V$$

and it remains to check that for such a V,

$$j_U^{-1}F_V \simeq i_{U*}F_V. \tag{11.1.2}$$

Recall that $j_U^{-1}F_V$ is the sheaf associated with $j_U^\dagger F_V$. For W open in X, one has by Proposition 8.3.8:

$$j_U^\dagger F_V(W) = \begin{cases} F_V(W) & \text{if } W \subset U, \\ 0 & \text{otherwise,} \end{cases}$$

and (11.1.2) follows. $\square$

Base change formula (non derived)

Consider a *Cartesian square* of locally compact topological spaces:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y. \end{tikzcd} \tag{11.1.3}$$

This means that $g \circ f' = f \circ g'$ and X' is isomorphic (as a topological space) to the fiber product:

$$X \times_Y Y' = \{(x, y') \in X \times Y';\ f(x) = g(y')\}.$$

Note that for any compact $K \subset Y'$, g' induces a topological isomorphism $f'^{-1}(K) \xrightarrow{\sim} f^{-1}(g(K))$.

Also note that choosing $y \in Y$ and setting $X' = f^{-1}(y)$, we get the Cartesian square:

$$\begin{tikzcd} f^{-1}(y) \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ \{y\} \arrow[r, "g"] & Y. \end{tikzcd} \tag{11.1.4}$$

Proposition 11.1.8. *Consider the Cartesian square* (11.1.3)*. There is a natural morphism of functors*

$$g^{-1} \circ f_* \to f'_* \circ g'^{-1}. \tag{11.1.5}$$

Moreover, if $F \in \mathrm{Mod}(\mathbf{k}_X)$ and f is proper on $\operatorname{supp} F$, then f' is proper on $\operatorname{supp} g'^{-1}F$ and the morphism (11.1.5) *induces an isomorphism $g^{-1}f_*F \xrightarrow{\sim} f'_*g'^{-1}F$. In particular, if f is proper, then* (11.1.5) *is an isomorphism.*

Proof. (i) The isomorphism $f_* \circ g'_* \simeq g_* \circ f'_*$ defines by adjunction the morphism $g^{-1} \circ f_* \circ g'_* \to f'_*$, hence the morphisms

$$\begin{aligned} g^{-1} \circ f_* &\to g^{-1} \circ f_* \circ g'_* \circ g'^{-1} \\ &\to f'_* \circ g'^{-1}. \end{aligned}$$

(ii) Let $y' \in Y'$ and set $y = g(y')$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. We have

$$\begin{aligned}(g^{-1}f_*F)_{y'} &\simeq (f_*F)_y \\ &\simeq \Gamma(f^{-1}(y);F)\end{aligned}$$

and

$$(f'_*g'^{-1}F)_{y'} \simeq \Gamma(f'^{-1}(y');g'^{-1}F).$$

Since g' induces a topological isomorphism $f'^{-1}(y') \xrightarrow{\sim} f^{-1}(g(y'))$, the result follows. □

Theorem 11.1.9. *Consider the Cartesian square* (11.1.3)*. Then there is a natural isomorphism of functors:*

$$f'_! \circ g'^{-1} \xrightarrow{\sim} g^{-1} \circ f_!.$$

In particular, setting $Y' = \{y\}$ for $y \in Y$, one gets the isomorphism for $F \in \mathrm{Mod}(\mathbf{k}_X)$:

$$(f_!F)_y \simeq \Gamma_c(f^{-1}(y);F|_{f^{-1}(y)}). \tag{11.1.6}$$

Proof. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. We have the isomorphisms below in which U ranges over the family of relatively compact open subsets of X and similarly with U' in X':

$$\begin{aligned}g^{-1}f_!F &\simeq g^{-1}\operatorname*{colim}_U f_*F_U \simeq \operatorname*{colim}_U g^{-1}f_*F_U \\ &\simeq \operatorname*{colim}_U f'_*(g'^{-1}(F_U)) \simeq \operatorname*{colim}_U f'_*((g'^{-1}F)_{g'^{-1}U})\end{aligned}$$

and

$$f'_!g'^{-1}F \simeq \operatorname*{colim}_{U'} f'_*(g'^{-1}F)_{U'}.$$

Let K be a compact subset of Y'. The family $\{f'^{-1}K \cap U'\}_{U'}$ and the family $\{f'^{-1}K \cap g^{-1}U\}_U$ are cofinal. Therefore, the morphism

$$\Gamma(K;f'_!g'^{-1}F) \to \Gamma(K;g^{-1}f_!F)$$

is an isomorphism. □

Projection formula (non derived)

Lemma 11.1.10. *Let X be a locally compact space and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Let M be a flat $\mathbf{k}$-module. Then the natural morphism:*

$$\Gamma_c(X;F) \otimes M \to \Gamma_c(X;F \otimes M_X)$$

is an isomorphism.

Proof. Since $\Gamma_c(X;F) \simeq \operatorname*{colim}_K \Gamma(X;F_K)$, we may assume from the beginning that X is compact. Let $K = \cup_j K_j$ be a finite covering by compact subsets and set $K_{ij} = K_i \cap K_j$. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X;F)\otimes M \arrow[r, "\lambda"] \arrow[d, "\alpha"'] & \oplus_i\Gamma(K_i;F)\otimes M \arrow[r, "\mu"] \arrow[d, "\beta"'] & \oplus_{ij}\Gamma(K_{ij};F)\otimes M \arrow[d, "\gamma"'] \\
0 \arrow[r] & \Gamma(X;F\otimes M_X) \arrow[r, "\lambda'"] & \oplus_i\Gamma(K_i;F\otimes M_X) \arrow[r, "\mu'"] & \oplus_{ij}\Gamma(K_{ij};F\otimes M_X)
\end{tikzcd}$$

Notice first the isomorphism

$$\operatorname*{colim}_U(\Gamma(U;F)\otimes M) \xrightarrow{\sim} \operatorname*{colim}_U \Gamma(U;F\otimes M_X), \tag{11.1.7}$$

where U ranges thourgh the family of open neighborhoods of $x \in X$. In fact, both sides are isomorphic to $F_x \otimes M$.

(i) α is injective. Let $s \in \Gamma(X;F)\otimes M$, with $\alpha(s) = 0$. By (11.1.7) there exists a covering such that $\lambda(s) = 0$. Hence, $s = 0$. The same argument shows that β and γ are injective.

(ii) α is surjective. Let $t \in \Gamma(X;F\otimes M_X)$. By (11.1.7) there exists a finite covering such that $\lambda'(t)$ is in the image of β. Then the result follows, using the injectivity of γ. □

Now we consider a continuous map $f\colon X \to Y$. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$. There are natural morphisms :

$$\begin{aligned} f^{-1}(f_*F\otimes G) &\simeq f^{-1}f_*F\otimes f^{-1}G \\ &\to F\otimes f^{-1}G \end{aligned}$$

which define by adjunction: $f_*F\otimes G \to f_*(F\otimes f^{-1}G)$. This last morphism induces:

$$f_!F\otimes G \to f_!(F\otimes f^{-1}G). \tag{11.1.8}$$

Proposition 11.1.11. *Let $f\colon X \to Y$ be a morphism of locally compact spaces. Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and $G \in \mathrm{Mod}(\mathbf{k}_Y)$. Assume that G is a flat $\mathbf{k}_Y$-module. Then the natural morphism (11.1.8) is an isomorphism.*

Proof. It is enough to check the isomorphism at each $y \in Y$. Denote by $g : \{y\} \hookrightarrow Y$ the embedding and consider the Cartesian square (11.1.4). Applying the base change formula, we get

$$\begin{aligned} (f_!(F\otimes f^{-1}G))_y &\simeq g^{-1}f_!(F\otimes f^{-1}G) \\ &\simeq f'_!g'^{-1}(F\otimes f^{-1}G) \\ &\simeq f'_!(g'^{-1}F\otimes g'^{-1}f^{-1}G). \end{aligned}$$

Applying Lemma 11.1.10 with F replaced by $g'^{-1}F$ and M replaced by $f^{-1}G = G_y$, we get

$$\begin{aligned} f'_!(g'^{-1}F\otimes g'^{-1}f^{-1}G) &\simeq f'_!g'^{-1}F\otimes G_y \\ &\simeq (f_!F)_y\otimes G_y \\ &\simeq (f_!F\otimes G)_y. \end{aligned}$$

□

11.2 c-soft sheaves

Definition 11.2.1. Assume X is locally compact. A sheaf F is c-soft if for any compact subset K of X, the map $\Gamma(X;F) \to \Gamma(K;F)$ is onto.

Proposition 11.2.2. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. The conditions bellow are equivalent:*

(i) *the sheaf F is c-soft*

(ii) *for any locally closed subset Z of X, the restriction map $\Gamma_c(X;F) \to \Gamma_c(Z;F|_Z)$ is surjective,*

(iii) *for any compact subset K of X, the restriction map $\Gamma_c(X;F) \to \Gamma(K;F)$ is surjective,*

Proof. (i) For K compact, we have $\Gamma(K;F) = \Gamma_c(K;F|_K)$. Therefore, $(ii) \Rightarrow (iii) \Rightarrow (i)$ is clear.

(ii) Assume F is c-soft. Let $s \in \Gamma_c(Z;F|_Z)$ with support in K and let U be a relatively compact open neighborhood of K in X. Define $\tilde{s} \in \Gamma(\partial U \cup (Z \cap \bar{U}); F)$ by setting $\tilde{s}|_{Z\cap\bar{U}} = s$, $\tilde{s}|_{\partial U} = 0$. Then $\tilde{s} \in \Gamma(\partial U \cup Z \cap \bar{U}; F)$ extends to a section of $\Gamma(X;F)$, and since $\tilde{s}|_{\partial U} = 0$, we may assume t is supported by $\bar{U}$. ☐

Corollary 11.2.3. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$ and let $M \in \mathrm{Mod}(\mathbf{k})$. Assume that F is c-soft and that M is flat. Then $F \otimes M_X$ is c-soft.*

Proof. This follows immediately from Proposition 11.2.2 and Lemma 11.1.10. ☐

Proposition 11.2.4. *A small filtered inductive limit of c-soft sheaves is c-soft. In particular, a small direct sum of c-soft sheaves is c-soft.*

Proof. The case of small filtered inductive limits follows from Propositions 11.1.7 and 11.2.2 and a small direct sum is a filtered inductive limit of finite direct sums. ☐

Proposition 11.2.5. *Assume F is c-soft on X.*

(i) *If $i_Z \colon Z \hookrightarrow X$ is the embedding of a locally closed subset in X, then $i_Z^{-1}F$ is c-soft,*

(ii) *If $f \colon X \to Y$ is continuous, then $f_!F$ is c-soft on Y,*

(iii) *for Z as in* (i)*, F_Z is c-soft.*

Proof. (i) If Z is open, this is clear and if Z is closed, this follows from Proposition 11.2.2.

(ii) Notice first that the equality $a_X = a_Y \circ f$ gives rise to the isomorphism $a_{X!} \simeq a_{Y!} \circ f_!$. Let K be a compact subset of Y. Consider the diagramin which the vertical arrows are isomorphisms:

$$\begin{tikzcd}
\Gamma_c(X;F) \arrow[r] \arrow[d, "\sim"] & \Gamma_c(f^{-1}(K);F) \arrow[d, "\sim"] \\
\Gamma_c(Y;f_!F) \arrow[r] & \Gamma_c(K;f_!F).
\end{tikzcd}$$

The top horizontal arrow is surjective by Proposition 11.2.2.

(iii) follows from (i) and (ii) since $F_Z \simeq i_{Z!}i_Z^{-1}F$. ☐

Proposition 11.2.6. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume F' is c-soft. Then the sequence*

$$0 \to \Gamma_c(X;F') \xrightarrow{\alpha} \Gamma_c(X;F) \xrightarrow{\beta} \Gamma_c(X;F'') \to 0$$

is exact.

Proof. Let $s'' \in \Gamma_c(X;F'')$ and let U be an open neighborhood of $\operatorname{supp}(s'')$, U being relatively compact. In order to prove that s is in the image of $\Gamma_c(X;F) \to \Gamma_c(X;F'')$, we may replace F', F, F'' by F'_U, F_U, F''_U. Then we may replace X by $\bar{U}$, hence we may assume from the beginning that X is compact.

Let $\{K_i\}_{i=1}^n$ be a finite covering of X by compact subsets and let $s_i \in \Gamma(K_i;F)$ such that $\beta(s_i) = s''|_{K_i}$. We argue by induction on n, and reduce the proof to the case $n = 2$. Then $s_1|_{K_1\cap K_2} - s_2|_{K_1\cap K_2}$ belongs to $\Gamma(K_1 \cap K_2; F')$. We extend this element to $s' \in \Gamma(X;F')$ and replace s_2 by $s_2 + s'$. Hence there exists $t \in \Gamma(K_1 \cup K_2; F)$ with $\beta(t) = s''$ and the induction proceeds. □

Proposition 11.2.7. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume F' and F are c-soft. Then F'' is soft.*

The proof is similar to that of Proposition 10.5.4.

Proposition 11.2.8. *Let S be a closed subset and K a compact subset of X. The category of c-soft sheaves is injective with respect to the functors $\Gamma_c(X;\bullet)$, $\Gamma_c(S;\bullet|_S)$, $f_!$ and $\Gamma(K;\bullet)$.*

The proof is left as an exercise.

Proposition 11.2.9. *Let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Then F is c-soft if and only if $\mathrm{R}\Gamma_c(U;F)$ is concentrated in degree 0 for all U open in X.*

Proof. (i) It follows from Proposition 11.2.8 that the condition is necessary.

(ii) Let us prove the converse. Let K be a compact subset. Applying Proposition 10.2.4, we have an exact sequence

$$0 \to F_{X\setminus K} \to F \to F_K \to 0.$$

Applying the functor $\Gamma_c(X;\bullet)$ to this exact sequence, the result follows since $H^1_c(X\setminus K;F) \simeq 0$ by the hypothesis. □

Corollary 11.2.10. *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves and assume F' and F'' are c-soft. Then F' is c-soft.*

Proof. Apply the functor $\Gamma_c(X;\bullet)$ to this exact sequence and use Proposition 11.2.9. □

Proposition 11.2.11. *Assume X is locally compact and countable at infinity. Then the category of c-soft sheaves is injective with respect to the functor $\Gamma(X;\bullet)$.*

Proof. Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves, with F' c-soft. Let $\{K_n\}_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of X, with $X = \cup_n K_n$.

The sequences

$$0 \to \Gamma(K_n;F') \to \Gamma(K_n;F) \to \Gamma(K_n;F'') \to 0$$

are all exact, and the morphisms $\Gamma(K_{n+1};F') \to \Gamma(K_n;F')$ are all surjective. Hence the sequence obtained by taking the projective limit will remain exact by the Mittag-Leffler property (Proposition 5.2.6). □

Proposition 11.2.12. *Assume X is locally compact and countable at infinity. Let $X = \bigcup_{i\in I} U_i$ be an open covering of X and let $F \in \mathrm{Mod}(\mathbf{k}_X)$. Assume that $F|_{U_i}$ is c-soft for all $i \in I$. Then F is c-soft.*

In other words, to be soft is a local property.

Proof. The proof is similar to that of Proposition 10.5.6. ☐

Proposition 11.2.13. [1] *Assume that $\mathbf{k}$ is a field. Let $F, K \in \mathrm{Mod}(\mathbf{k}_X)$ and assume that F is c-soft. Then $K \otimes F$ is c-soft.*

Proof. (i) There exists a small family of open sets $\{U_i\}_{i\in I}$ and an epimorphism $\bigoplus_{i\in I} \mathbf{k}_{U_i} \twoheadrightarrow K$. For a finite subset J of I, denote by K_J the image of $\bigoplus_{j\in J} \mathbf{k}_{U_j}$, a subsheaf of K. Then $K \simeq \mathrm{colim}\, K_J$ and it is enough to prove that the K_J's are c-soft by Proposition 11.2.4. Then the proof goes by induction on the cardinal of J.

(ii) If $|J| = 1$, then $K \simeq \mathbf{k}_Z$ for a locally closed subset Z of X by the result of Exercise 10.13 and, in this case, the result follows from Proposition 11.2.5.

(iii) Assume that the result is proved for $|J| < N$ and let us write $J = J' \sqcup J''$ for non empty J' and J''. Define $K_{J'}$ as the subsheaf of K_J, the image of $\bigoplus_{j\in J'} \mathbf{k}_{U_j}$ and similarly with $K_{J''}$. There is an epimorphism $\varphi\colon K_{J'} \oplus K_{J''} \twoheadrightarrow K_J$. Define K'' by the exact sequence $0 \to K_{J'} \to K_J \to K'' \to 0$. Then φ induces an epimorphism $K_{J''} \twoheadrightarrow K''$. Therefore, K'' is c-soft and the induction proceeds by Corollary 11.2.10. ☐

Example 11.2.14. (i) On a locally compact space X, any sheaf of C^0_X-modules is c-soft.

(ii) Let X be a real manifold of class C^∞, let K be a compact subset of X and U an open neighborhood of K in X. By the existence of "partition of unity", there exists a real $\mathscr{C}^\infty$-function φ with compact support containc-softed in U and which is identically 1 in a neighborhood of K. It follows that any sheaf of C^∞_X-modules is c-soft.

(iii) Flabby sheaves are c-soft.

11.3 Derived proper direct images

Consider a morphism $f\colon X \to Y$ of locally compact spaces. One denotes by $\mathrm{R}f_!$ its right derived functor:

$$\mathrm{R}f_!\colon \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y).$$

By Proposition 11.2.8, if $F \in \mathrm{Mod}(\mathbf{k}_X)$, then $\mathrm{R}f_!F \simeq f_!F^\bullet$, where $F^\bullet$ is a c-soft resolution of F. Moreover, if $g\colon Y \to Z$ is another morphism of locally compact spaces, then, by Proposition 11.2.5 (ii),

$$\mathrm{R}(g \circ f)_! \simeq \mathrm{R}g_! \circ \mathrm{R}f_!. \tag{11.3.1}$$

Definition 11.3.1. Let $d \in \mathbb{N}$. One says that f has c-soft dimension $\leq d$ if $H^j(\mathrm{R}f_!F) = 0$ for all $j > d$ and all $F \in \mathrm{Mod}(\mathbf{k}_X)$. One says that f has finite c-soft dimension if there exists $d \geq 0$ such that f has c-soft dimension $\leq d$.

---

[1] This result is a variation on a theorem of [KS01, Prop. 4.2.21]

In the sequel, we shall always make Hypothesis 11.3.2 below.

Hypothesis 11.3.2. The map f has finite c-soft dimension.

Remark 11.3.3. It follows from Theorem 11.1.2 that f has c-soft dimension $\leq d$ only if, for any $y \in Y$, the restriction $f|_{f(y)^{-1}}$ has c-soft dimension $\leq d$.

Note that assuming Hypothesis 11.3.2, the functor $\mathrm{R}f_!$ induces a functor:

$$\mathrm{R}f_!\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X) \to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y).$$

Projection formula

First, we shall obtain a derived version of the isomorphism in Proposition 11.1.11.

Theorem 11.3.4. (Projection formula.) *Let $f\colon X \to Y$ be a morphism of locally compact spaces. Let $F \in \mathrm{D}^+(\mathbf{k}_X)$ and $G \in \mathrm{D}^+(\mathbf{k}_Y)$. Then there is a natural isomorphism in $\mathrm{D}^+(\mathbf{k}_Y)$*

$$\mathrm{R}f_!F\overset{\mathrm{L}}{\otimes}G \simeq \mathrm{R}f_!(F\overset{\mathrm{L}}{\otimes}f^{-1}G).$$

Proof. Let $F^\bullet$ be a c-soft resolution of F in $\mathrm{K}^+(\mathrm{Mod}(\mathbf{k}_X))$ and let $G^\bullet$ be a flat resolution of G in $\mathrm{K}^+(\mathrm{Mod}(\mathbf{k}_Y))$ (such a bounded resolution exists thanks to the hypothesis on $\mathbf{k}$).

It follows from Corollary 11.2.3 and (11.1.6) that, if F^i is c-soft and G^j is flat, then $F^i \otimes f^{-1}G^j$ is acyclic for the functor $f_!$. Therefore, $\mathrm{R}f_!(F\overset{\mathrm{L}}{\otimes}f^{-1}G)$ is represented by the complex $f_!(F^\bullet \otimes f^{-1}G^\bullet)$. On the other hand, $\mathrm{R}f_!F\overset{\mathrm{L}}{\otimes}G$ is represented by the complex $f_!F^\bullet \otimes G^\bullet$. Hence, the result follows from Proposition 11.1.11. ☐

Base change formula

Next, we shall obtain a derived version of the isomorphism in Theorem 11.1.9.

Theorem 11.3.5. (Base change formula.) *Consider the Cartesian square (11.1.3). Then there is an isomorphism in $\mathrm{D}^+(\mathbf{k}_{Y'})$, functorial in $F \in \mathrm{D}^+(\mathbf{k}_X)$:*

$$g^{-1}\mathrm{R}f_!F \simeq \mathrm{R}f'_!g'^{-1}F.$$

In particular, for $y \in Y$, we have the isomorphism

$$(\mathrm{R}f_!F)_y \simeq \mathrm{R}\Gamma_c(f^{-1}(y); F|_{f^{-1}(y)}). \tag{11.3.2}$$

Proof. It is enough to prove that $g^{-1} \circ \mathrm{R}f_!$ is the derived functor of $g^{-1} \circ f_!$, which is obvious since g^{-1} is exact and $\mathrm{R}f_! \circ g'^{-1}$ is the derived functor of $f'_! \circ g'^{-1}$.

Denote by $\mathscr{I}_X$ the subcategory of $\mathrm{Mod}(\mathbf{k}_X)$ consisting of sheaves F such that for all $y \in Y$, $F|_{f^{-1}(y)}$ is c-soft, and define similarly $\mathscr{I}_{X'}$. Then $\mathscr{I}_X$ is injective with respect to g'^{-1} and g'^{-1} sends $\mathscr{I}_X$ into $\mathscr{I}_{X'}$. Moreover, $\mathscr{I}_{X'}$ is injective with respect to $f'_!$. ☐

Let us give two important corollaries. The first one tells us that the cohomology with compact support of a topological space X, with values in a commutative group M, *i.e.*, the cohomology of the constant sheaf M_X, is known as soon as it is known over $\mathbb{Z}$. The second one tells us how to calculate the cohomology of a product.

Universal coefficients formula

Corollary 11.3.6. (Universal coefficients formula.) *Let* $M \in \mathrm{D^b}(\mathbf{k})$.

(i) *One has the isomorphism* $\mathrm{R}\Gamma_c(X; M_X) \simeq \mathrm{R}\Gamma_c(X; \mathbf{k}_X)\overset{\mathrm{L}}{\otimes}M$.

(ii) *Assume* $\mathbf{k} = \mathbb{Z}$. *Then*

$$\begin{aligned}\mathrm{R}\Gamma_c(X; M_X) &\simeq \bigoplus_j H^j_c(X; M_X)\,[-j]\\ &\simeq \bigoplus_j \Big(H^j_c(X; \mathbb{Z}_X)\otimes_{\mathbb{Z}} M \oplus \mathrm{Tor}_1^{Z}(H^{j+1}_c(X; \mathbb{Z}_X), M)\Big)\,[-j].\end{aligned}$$

Proof. (i) One has $M_X = a_X^{-1}M_{\mathrm{pt}}\overset{\mathrm{L}}{\otimes}\mathbf{k}_X$. By the projection formula, we get:

$$\mathrm{R}a_{X!}(a_X^{-1}M_{\mathrm{pt}}\overset{\mathrm{L}}{\otimes}\mathbf{k}_X) \simeq \mathrm{R}a_{X!}\mathbf{k}_X\overset{\mathrm{L}}{\otimes}M.$$

(ii) Since the homological dimension of the ring $\mathbb{Z}$ is one, we have for $N \in \mathrm{D^b}(\mathbb{Z})$ and $M \in \mathrm{Mod}(\mathbb{Z})$:

$$\begin{aligned}&N \simeq \oplus_j H^j(N)[-j],\\ &N\overset{\mathrm{L}}{\otimes}_{\mathbb{Z}}M \simeq \bigoplus_j \big(H^j(N)\otimes M \oplus Tor_1^{\mathbb{Z}}(H^{j+1}(N), M)\big)\,[-j].\end{aligned}$$

□

Notation 11.3.7. Let X and Y be two topological spaces. Let $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G \in \mathrm{D}^+(\mathbf{k}_Y)$. One sets:

$$F\overset{\mathrm{L}}{\boxtimes}G = q_1^{-1}F\overset{\mathrm{L}}{\otimes}q_2^{-1}G.$$

Künneth formula

Corollary 11.3.8. (Künneth formula.) *Let X and Y be two locally compact spaces satisfying* (11.3.2). *Let* $F \in \mathrm{D}^*(\mathbf{k}_X)$, $G \in \mathrm{D}^*(\mathbf{k}_Y)$ *with* $* = \mathrm{b}, +, -$. *Then:*

$$\mathrm{R}\Gamma_c(X\times Y; F\overset{\mathrm{L}}{\boxtimes}G) \simeq \mathrm{R}\Gamma_c(X; F)\overset{\mathrm{L}}{\otimes}\mathrm{R}\Gamma_c(Y; G). \tag{11.3.3}$$

Proof. Consider the Cartesian square:

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X \arrow[dr, "a_X"'] & & Y \arrow[dl, "a_Y"] \\ & \mathrm{pt} & \end{tikzcd} \tag{11.3.4}$$

Then:

$$\begin{aligned}
\mathrm{R}a_{X\times Y!}(F \overset{\mathrm{L}}{\boxtimes} G) &\simeq \mathrm{R}a_{Y!}\mathrm{R}p_{2!}(p_1^{-1}F \overset{\mathrm{L}}{\otimes} p_2^{-1}G)\\
&\simeq \mathrm{R}a_{Y!}((\mathrm{R}p_{2!}p_1^{-1}F)\overset{\mathrm{L}}{\otimes}G)\\
&\simeq \mathrm{R}a_{Y!}(a_Y^{-1}\mathrm{R}a_{X!}F\overset{\mathrm{L}}{\otimes}G)\\
&\simeq \mathrm{R}a_{X!}F\overset{\mathrm{L}}{\otimes}\mathrm{R}a_{Y!}G.
\end{aligned}$$

We have applied Theorem 11.3.4 in the second and fourth isomorphisms and Theorem 11.3.5 in the third one. ☐

Remark 11.3.9. We have defined the functor $\mathrm{R}f_!$ for a morphism $f\colon X \to Y$ of locally compact topological spaces. In algebraic geometry, the spaces one encounters are not of this nature. However, if X is a site and $U \in \mathscr{C}_X$, the functor $i_{U!}$ is well-defined (see Notation 8.7.9).

Now let $f\colon X \to Y$ be a morphism of schemes and let us say that f is compactifiable if there exists a commutative diagram of schemes

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i_X"] & Y \arrow[d, "i_Y"] \\
\widetilde{X} \arrow[r, "\widetilde{f}"'] & \widetilde{Y}
\end{tikzcd}$$

such that the morphism $\widetilde{f}$ is proper (we do not define precisely here what means proper for schemes). One then defines the functor $\mathrm{R}f_!$ as the composition $i_Y^{-1} \circ \mathrm{R}\widetilde{f}_* \circ i_{X!}$. Morphisms of schemes are compactifiable under very general hypotheses. See [Del66].

11.4 The functor $f^!$

All over this section, we shall assume that all morphisms of locally compact spaces have finite c-soft dimension (see hypothesis 11.3.2).

Thanks to Proposition 11.1.7 and Lemma 11.2.4, we may apply Theorem 7.5.7 to the functor $f_!$. We get:

Theorem 11.4.1. *Let $f\colon X \to Y$ be a continuous map of locally compact topological spaces satisfying (11.3.2). Then the functor $\mathrm{R}f_! : \mathrm{D}^+(\mathbf{k}_X) \to \mathrm{D}^+(\mathbf{k}_Y)$ admits a right adjoint.*

One denotes by $f^!$ this adjoint.

In other words, for $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G \in \mathrm{D}^+(\mathbf{k}_Y)$, we have an isomorphism functorial with respect to F and G:

$$\mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_Y)}(\mathrm{R}f_!F, G) \simeq \mathrm{Hom}_{\mathrm{D}^+(\mathbf{k}_X)}(F, f^!G). \tag{11.4.1}$$

Notice that the functor $f^!\colon \mathrm{D}^+(\mathbf{k}_Y) \to \mathrm{D}^+(\mathbf{k}_X)$ is not the derived functor of any functor in general. However, we have already encountered this functor in case of an open embedding.

Proposition 11.4.2. *Let $i_U\colon U \hookrightarrow X$ be an open embedding. Then $i_U^! \simeq i_U^{-1}$.*

Proof. Both functors are right adjoint to the functor $i_{U!}$ by Propositions 11.1.7 and 8.7.8. ☐

For a direct proof of Theorem 11.4.1, not using the Brown representability theorem, we refer to [Ver69, GM96, KS90].

We discuss its applications. First, notice that we get natural morphisms:

$$\mathrm{R}f_!f^!G \to G, \quad F \to f^!\mathrm{R}f_!F.$$

Proposition 11.4.3. *Let $g\colon Y \to Z$ be a continuous map satisfing Hypothesis 11.3.2. Then $g \circ f$ satisfies Hypothesis 11.3.2 and*

$$(g \circ f)^! \simeq f^! \circ g^!.$$

Proof. Both results immediately follow from (11.3.1). ☐

Proposition 11.4.4. *Consider the Cartesian square (11.1.3) Assume f satisfies Hypothesis 11.3.2. Then f' satisfies Hypothesis 11.3.2 and there is a natural isomorphism of functors from $\mathrm{D}^+(\mathbf{k}_{X'})$ to $\mathrm{D}^+(\mathbf{k}_Y)$:*

$$f^! \circ \mathrm{R}g_* \simeq \mathrm{R}g'_* \circ f'^!. \tag{11.4.2}$$

Proof. The results follow from Theorem 11.3.5 by adjunction. ☐

Proposition 11.4.5. *In the situation of Theorem 11.4.1, one has for $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$ and $G \in \mathrm{D}^+(\mathbf{k}_Y)$:*

(a) $\mathrm{RHom}\,(F, f^!G) \xrightarrow{\sim} \mathrm{RHom}\,(\mathrm{R}f_!F, G)$ *in* $\mathrm{D}^+(\mathbf{k})$,

(b) $\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F, f^!G) \xrightarrow{\sim} \mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F, G)$ *in* $\mathrm{D}^+(\mathbf{k}_Y)$.

Proof. (i) The sequence of morphisms

$$\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F, f^!G) \to \mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F, \mathrm{R}f_!f^!G) \to \mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F, G)$$

defines the morphism in (b). Applying the functor $\mathrm{R}a_{Y*}$ and using Theorem 9.3.3, we get the morphism in (a).

(ii) To prove that the morphism in (a) is an isomorphism, it is enough to prove that it induces an isomorphism when applying the cohomology functor H^j. This follows from (11.4.1) by Theorem 7.4.9.

(iii) To prove that the morphism in (b) is an isomorphism, we shall apply $\mathrm{R}\Gamma(V; \bullet)$ to both terms for V open in Y. Set $U := f^{-1}(V)$ and denote by f_V the restriction of f to U. Hence, we have a Cartesian square:

$$\begin{tikzcd} U \arrow[r, "i_U"] \arrow[d, "f_V"] & X \arrow[d, "f"] \\ V \arrow[r, "i_V"] & Y. \end{tikzcd}$$

Recall that a_X is the map $X \to \mathrm{pt}$. Applying Theorem 11.3.5 (and using Proposition 11.4.2), we get:

$$\begin{aligned}\mathrm{R}\Gamma(V;\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F,f^!G)) &\simeq \mathrm{R}a_{V*}i_V^{-1}\mathrm{R}f_*\mathrm{R}\mathscr{H}om\,(F,f^!G))\\ &\simeq \mathrm{R}a_{V*}\mathrm{R}f_{V*}i_U^{-1}\mathrm{R}\mathscr{H}om\,(F,f^!G))\\ &\simeq \mathrm{R}a_{U*}\mathrm{R}\mathscr{H}om\,(i_U^{-1}F,i_U^{-1}f^!G))\\ &\simeq \mathrm{RHom}\,(i_U^{-1}F,f_V^!i_V^{-1}G)\\ &\simeq \mathrm{RHom}\,(\mathrm{R}f_{V!}i_U^{-1}F,i_V^{-1}G)\\ &\simeq \mathrm{R}\Gamma(V;\mathrm{R}\mathscr{H}om\,(\mathrm{R}f_!F,G)).\end{aligned}$$

□

Proposition 11.4.6. *Let* $G_1, G_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$. *There is a natural morphism in* $\mathrm{D}^+(\mathbf{k}_X)$:

$$f^!G_1 \otimes f^{-1}G_2 \to f^!(G_1\overset{\mathrm{L}}{\otimes}G_2).$$

Proof. We shall apply Theorem 11.3.4. Consider the chain of morphisms:

$$\begin{aligned}\mathrm{Hom}\,(G_1\overset{\mathrm{L}}{\otimes}G_2,H) &\to \mathrm{Hom}\,(\mathrm{R}f_!f^!G_1\overset{\mathrm{L}}{\otimes}G_2,H)\\ &\simeq \mathrm{Hom}\,(\mathrm{R}f_!(f^!G_1\overset{\mathrm{L}}{\otimes}f^{-1}G_2),H)\\ &\simeq \mathrm{Hom}\,(f^!G_1\overset{\mathrm{L}}{\otimes}f^{-1}G_2,f^!H).\end{aligned}$$

Choosing $H = G_1 \otimes G_2$, we get the result. □

Given a map $f\colon X \to Y$, we may decompose it by its graph:

$$f\colon X \hookrightarrow X \times Y \to Y.$$

In view of Proposition 11.4.3, in order to calculate $f^!$ it is thus enough to do it when f is a closed embedding and when f is a projection.

Proposition 11.4.7. *Assume that* $f\colon X \to Y$ *is a closed embedding, hence, induces an isomorphism from* X *onto a closed subset* Z *of* Y. *Then*

$$f^!(\,\bullet\,) \simeq f^{-1} \circ \mathrm{R}\Gamma_Z(\,\bullet\,).$$

Proof. Let $F \in \mathrm{D}^+(\mathbf{k}_X)$, $G \in \mathrm{D}^+(\mathbf{k}_Y)$.

$$\begin{aligned}\mathrm{Hom}\,(F,f^!G) &\simeq \mathrm{Hom}\,(\mathrm{R}f_!F,G)\\ &\simeq \mathrm{Hom}\,(\mathrm{R}f_!F\otimes\mathbf{k}_Z,G) \simeq \mathrm{Hom}\,(\mathrm{R}f_!F,\mathrm{R}\Gamma_ZG)\\ &\simeq \mathrm{Hom}\,(f^{-1}\mathrm{R}f_!F,f^{-1}\mathrm{R}\Gamma_ZG) \simeq \mathrm{Hom}\,(F,f^{-1}\mathrm{R}\Gamma_ZG).\end{aligned}$$

Since these isomorphisms are functorial with respect to $F \in \mathrm{D}^+(\mathbf{k}_X)$, the result follows. □

Proposition 11.4.8. *Let* $G_1, G_2 \in \mathrm{D}^+(\mathbf{k}_Y)$. *Then:*

$$f^!\mathrm{R}\mathscr{H}om\,(G_2,G_1) \simeq \mathrm{R}\mathscr{H}om\,(f^{-1}G_2,f^!G_1).$$

Proof. For $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, one has:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)}(F, f^!\mathrm{R}\mathscr{H}om(G_2, G_1)) &\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)}(\mathrm{R}f_!F, \mathrm{R}\mathscr{H}om(G_2, G_1)) \\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)}(\mathrm{R}f_!F \overset{\mathrm{L}}{\otimes} G_2, G_1) \\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)}(\mathrm{R}f_!(F \overset{\mathrm{L}}{\otimes} f^{-1}G_2), G_1) \\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)}(F \overset{\mathrm{L}}{\otimes} f^{-1}G_2, f^!G_1) \\
&\simeq \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)}(F, \mathrm{R}\mathscr{H}om(f^{-1}G_2, f^!G_1)).
\end{aligned}$$

Since these isomorphisms are functorial with respect to $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$, the result follows. □

Consider the diagram, were as usual δ_X denotes the diagonal embedding:

$$\begin{tikzcd}
\Delta_X \arrow[r, hook, "\delta"] & X \times X \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\
X & & X
\end{tikzcd}$$

Corollary 11.4.9. *Let $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then, identifying Δ_X with X by q_1,*

$$\mathrm{R}\mathscr{H}om(F_2, F_1) \simeq \delta^!\mathrm{R}\mathscr{H}om(q_2^{-1}F_2, q_1^!F_1).$$

Proof.

$$\begin{aligned}
\delta^!\mathrm{R}\mathscr{H}om(q_2^{-1}F_2, q_1^!F_1) &\simeq \mathrm{R}\mathscr{H}om(\delta^{-1}q_2^{-1}F_2, \delta^!q_1^!F_1) \\
&\simeq \mathrm{R}\mathscr{H}om(F_2, F_1).
\end{aligned}$$

□

The next proposition is analogous to the Künneth formula, replacing the functor $q_2^{-1}(\,\bullet\,)\overset{\mathrm{L}}{\otimes}q_1^!(\,\bullet\,)$ with the functor $\mathrm{RHom}(q_2^{-1}(\,\bullet\,), q_1^!(\,\bullet\,))$.

Proposition 11.4.10. *Let X and Y be topological spaces with finite c-soft dimension. Then for $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_Y)$, $F \in \mathrm{D}^{+}(\mathbf{k}_X)$, one has:*

$$\mathrm{RHom}(q_2^{-1}G, q_1^!F)) \simeq \mathrm{RHom}(\mathrm{R}\Gamma_c(Y; G), \mathrm{R}\Gamma(X; F)).$$

Proof. Consider Diagram 11.3.4. Then:

$$\begin{aligned}
\mathrm{R}\Gamma(X \times Y; \mathrm{R}\mathscr{H}om(q_2^{-1}G, q_1^!F)) &\simeq \mathrm{R}a_{X*}\mathrm{R}q_{1*}\mathrm{R}\mathscr{H}om(q_2^{-1}G, q_1^!F) \\
&\simeq \mathrm{R}a_{X*}\mathrm{R}\mathscr{H}om(\mathrm{R}q_{1!}q_2^{-1}G, F) \\
&\simeq \mathrm{R}a_{X*}\mathrm{R}\mathscr{H}om(a_X^{-1}\mathrm{R}a_{Y!}G, F) \\
&\simeq \mathrm{RHom}(\mathrm{R}a_{Y!}G, \mathrm{R}a_{X*}F).
\end{aligned}$$

□

Definition 11.4.11. Assume that $f\colon X \to Y$ satisfies Hypothsis 11.3.2. One sets:

$$\omega_{X/Y} := f^!\mathbf{k}_Y$$

and calls $\omega_{X/Y}$ the relative dualizing complex.

If X has finite c-soft dimension, one sets:

$$\omega_X = \omega_{X/\mathrm{pt}} = f^!\mathbf{k}_{\mathrm{pt}},$$

and calls ω_X the dualizing complex on X.

Note that by applying Proposition 11.4.6 with $F_1 = \mathbf{k}_X$, we get a natural morphism:

$$f^{-1}G \otimes \omega_{X/Y} \to f^!G.$$

For $F \in \mathrm{D^b}(\mathbf{k}_X)$, one defines the two dual objects to F:

$$\begin{aligned} \mathrm{D}'_X F &= \mathrm{R}\mathscr{H}om\,(F, \mathbf{k}_X), \\ \mathrm{D}_X F &= \mathrm{R}\mathscr{H}om\,(F, \omega_X). \end{aligned}$$

We denote by * the duality functor on $\mathrm{Mod}(\mathbf{k})$:

$$^* = \mathrm{RHom}_{\mathbf{k}}(\,\bullet\,, \mathbf{k}),\ \mathrm{D^b}(\mathrm{Mod}(\mathbf{k}))^{\mathrm{op}} \to \mathrm{D}^+(\mathrm{Mod}(\mathbf{k})). \tag{11.4.3}$$

Using the adjunction $(\mathrm{R}a_{X!}, a_X^!)$, we get :

$$\begin{aligned} \mathrm{RHom}\,(F, \omega_X) &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(X; F), \mathbf{k}) \\ &= (\mathrm{R}\Gamma_c(X, F))^*. \end{aligned}$$

Choosing $F := \mathbf{k}_X$, we find:

Corollary 11.4.12. *Assume that X has finite c-soft dimension. Then*

$$(\mathrm{R}\Gamma_c(X; \mathbf{k}_X))^* \simeq \mathrm{R}\Gamma(X; \omega_X).$$

When X is a topological n-dimensional manifold of class $\mathscr{C}^\infty$, we shall see that ω_X is the orientation sheaf shifted by n, and Corollary 11.4.12 is a formulation of the classical Poincaré duality theorem.

11.5 Orientation and duality on $\mathscr{C}^0$-manifolds

A $\mathscr{C}^0$-manifold X is a Hausdorff, locally compact, countable at infinity topological space which is locally isomorphic to a real finite dimensional vector space. Recall that the dimension of such a vector space is a topological invariant, hence the dimension of X is a well-defined locally constant function on X that we denote by d_X.

Lemma 11.5.1. *Let V be a real vector space of dimension n and let F be a sheaf on V. Then $H^j_c(V; F) = 0$ for $j > n$.*

Proof. (i) Assume $n = 1$. We may replace V by the open interval $]0,1[$. Denote by j the embedding $]0,1[\hookrightarrow [0,1]$. Then $j_!$ is exact and we deduce that $H^j_c(]0,1[;F) \simeq H^j([0,1];j_!F)$. Then, the result follows from (11.3.1) and Lemma 10.6.1.

(ii) Assume the result if proved for linear spaces of dimension less than n. Let $p: V \to V'$ be a surjective linear map with $\dim V' = n-1$. By the above result and the base change formula, $R^jp_! = 0$ for $j \neq 0, 1$. Hence have a d.t. $R^0p_!F \to \mathrm{R}p_!F \to R^1p_!F[-1] \xrightarrow{+1}$, which gives a long exact sequence:

$$\cdots \to H^j_c(V';R^0p_!F) \to H^j_c(V;F) \to H^{j-1}_c(V';R^1p_!F) \to \cdots$$

Then the result follows by induction. ☐

Proposition 11.5.2. *Let X be a $\mathscr{C}^0$-manifold of constant dimension n and let F be a sheaf on X. Then:*

(i) $H^j(X;F) = 0$ *for* $j > n$,

(ii) $H^j_c(X;F) = 0$ *for* $j > n$,

(iii) *the c-soft dimension of X is n.*

Proof. (i)–(ii) Let $0 \to F \to F^0 \xrightarrow{d^0} F^1 \xrightarrow{d^1} \cdots$ be an injective resolution of F, and let $G^n := \ker d^n$. It is enough to prove that G^n is c-soft. This is a local problem, and we may assume $X = V$ is a real vector space. Let U be an open subset of V. Since $H^j_c(U;F) \simeq H^j_c(V;F_U)$, these groups vanish for $j > n$ by Lemma 11.5.1 and the result follows from Proposition 11.2.9.

(iii) By (ii) the c-soft dimension of X is $\leq n$. The result follows since $H^n_c(X;\mathbf{k}_X) \neq 0$ when $X = \mathbb{R}^n$. ☐

Lemma 11.5.3. *Let X be a topological manifold of dimension n. Then $H^k(\omega_X) = 0$ for $k \neq -n$, and the sheaf $H^{-n}(\omega_X)$ is locally isomorphic to $\mathbf{k}_X$.*

Proof. We may assume $X = \mathbb{R}^n$. Then for U open in X, one has the isomorphisms:

$$\begin{aligned} \mathrm{R}\Gamma(U;\omega_X) &\simeq \mathrm{RHom}\,(\mathbf{k}_U, a_X^!\mathbf{k}) \\ &\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbf{k}_X),\mathbf{k}) \\ &= (\mathrm{R}\Gamma_c(U;\mathbf{k}_X))^*. \end{aligned}$$

If U is convex and non empty, one already knows that $\mathrm{R}\Gamma_c(U;\mathbf{k}_U)$ is isomorphic to $\mathbf{k}\,[-n]$. Hence $H^k(\omega_X) = 0$ for $k \neq -n$ and the restriction morphisms $\Gamma(X;H^{-n}(\omega_X)) \to \Gamma(U;H^{-n}(\omega_X))$ are isomorphisms for U convex and non empty. ☐

Definition 11.5.4. Let X be a $\mathscr{C}^0$-manifold of dimension d_X. One sets:

$$\mathrm{or}^{\mathbf{k}}_X = H^{-d_X}(\omega_X)$$

and calls this sheaf the orientation sheaf on X. If there is no risk of confusion, we write or_X instead of $\mathrm{or}^{\mathbf{k}}_X$.

Note that

$$\omega_X \simeq \mathrm{or}^{\mathbf{k}}_X[d_X], \quad \mathrm{or}^{\mathbf{k}}_X \simeq \mathrm{or}^{\mathbb{Z}}_X \otimes_{\mathbb{Z}_X} \mathbf{k}_X.$$

Proposition 11.5.5. *Let X be a $\mathscr{C}^0$-manifold of dimension d_X.*

(i) or_X *is the sheaf associated to the presheaf:* $U \mapsto \mathrm{Hom}_k(H_c^{d_X}(U;\mathbf{k}_X)),\mathbf{k})$,

(ii) or_X *is locally free of rank one over* $\mathbf{k}_X$*, and* $\mathrm{or}_{X,x} \simeq (H^{d_X}_{\{x\}}(\mathbf{k}_X))^*$,

(iii) $\mathrm{or}_X \otimes \mathrm{or}_X \simeq \mathbf{k}_X$*, and* $\mathscr{H}om\,(\mathrm{or}_X,\mathbf{k}_X) \simeq \mathrm{or}_X$,

(iv) *if X is of class $\mathscr{C}^1$, then or_X coincides with the orientation sheaf defined in* Example 8.8.4.

Assertions (i) to (iii) are easily deduced from the previous discussion. We refer to [KS90] for a proof of (iv).

Applying Corollary 11.4.12, we obtain the Poincaré duality theorem with coefficients in $\mathbf{k}$:

Corollary 11.5.6. (Poincaré duality.) *Let X be $\mathscr{C}^0$-manifold of dimension d_X. Then*

$$(\mathrm{R}\Gamma_c(X;\mathbf{k}_X))^* \simeq \mathrm{R}\Gamma(X;\mathrm{or}_X)\,[d_X].$$

Definition 11.5.7. Let $f\colon X \to Y$ be a continuous map of topological spaces. One says that f is a topological submersion of relative dimension d if, locally on X, there exists an isomorphism $X \simeq Y \times \mathbb{R}^d$ and a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[d, "f"'] & Y \times \mathbb{R}^d \arrow[dl, "p"] \\
Y &
\end{tikzcd}$$

such that p is the projection.

Proposition 11.5.8. *Assume that $f\colon X \to Y$ is a topological submersion of relative dimension d. Let $G \in \mathrm{D}^+(\mathbf{k}_Y)$. Then there is a natural isomorphism $f^{-1}G\overset{\mathrm{L}}{\otimes}\omega_{X/Y} \xrightarrow{\sim} f^!G$.*

Proof. The natural morphism $f^{-1}G\overset{\mathrm{L}}{\otimes}\omega_{X/Y} \to f^!G$ is given by Proposition 11.4.6. To check it is an isomorphism, we may assume $X = Y \times T$ and f is the projection. We may assume $Y = U$ is a non empty open convex subset of a real vector space of dimension d. Then

$$\begin{aligned}
\mathrm{R}\Gamma(U\times V; f^!G) &\simeq \mathrm{RHom}\,(\mathbf{k}_{U\times V}, f^!G)\\
&\simeq \mathrm{RHom}\,(\mathrm{R}f_!\mathbf{k}_{U\times V}, G)\\
&\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbf{k}_U)\overset{\mathrm{L}}{\otimes}\mathbf{k}_V, G)\\
&\simeq \mathrm{RHom}\,(\mathrm{R}\Gamma_c(U;\mathbf{k}_U),\mathbf{k}) \otimes \mathrm{RHom}\,(\mathbf{k}_V, G)\\
&\simeq \mathrm{RHom}\,(\mathbf{k}_V, G)[d].
\end{aligned}$$

Here, we use the fact that the cohomology of $\mathrm{R}\Gamma_c(U;\mathbf{k}_U)$ is isomorphic to $\mathbf{k}[-d]$. On the other hand, since $\omega_{X/Y}$ is locally isomorphic to $\mathbf{k}_X[d]$, it remains to remark that

$$\mathrm{R}\Gamma(U\times V; f^{-1}G) \simeq \mathrm{R}\Gamma(V;G).$$

□

11.6 Cohomology of real and complex manifolds

De Rham cohomology

Let X be a real $\mathscr{C}^\infty$-manifold of dimension n (this implies in particular that X is locally compact and countable at infinity). If $n > 0$, the sheaf $\mathbb{C}_X$ is not acyclic for the functor $\Gamma(X;\,\bullet\,)$ in general. In fact consider two connected open subsets U_1 and U_2 such that $U_1 \cap U_2$ has two connected components, V_1 and V_2. The sequence:

$$0 \to \Gamma(U_1 \cup U_2; \mathbb{C}_X) \to \Gamma(U_1; \mathbb{C}_X) \oplus \Gamma(U_2; \mathbb{C}_X) \to \Gamma(U_1 \cap U_2; \mathbb{C}_X) \to 0$$

is not exact since the locally constant function $\varphi = 1$ on V_1, $\varphi = 2$ on V_2 may not be decomposed as $\varphi = \varphi_1 - \varphi_2$, with φ_j constant on U_j. By the Mayer-Vietoris long exact sequence, this implies:

$$H^1(U_1 \cup U_2; \mathbb{C}_X) \neq 0.$$

On the other hand, for K a compact subset in X and U an open neighborhood of K in X, there exists a real $\mathscr{C}^\infty$-function φ with compact support contained in U and which is identically 1 in a neighborhood of K (existence of "partition of unity"). This implies that the sheaf $\mathscr{C}^\infty_X$ is c-soft, as well as any sheaf of $\mathscr{C}^\infty_X$-modules. In particular, the sheaves $\mathscr{C}^{\infty,(p)}_X$ or $\mathscr{D}b^{(p)}_X$ of differential forms with $\mathscr{C}^\infty_X$ or distributions coefficients are c-soft and in particular $\Gamma(X;\cdot)$ and $\Gamma_c(X;\cdot)$ acyclic.

Recall that by its definition, the space $\Gamma_c(X;\mathscr{D}b_X)$ of distributions with compact support is the topological dual of the space $\Gamma(X;\mathscr{C}^{\infty,(n)}_X \otimes \mathrm{or}_X)$ of $\mathscr{C}^\infty$-densities. Integration over X defines the embedding of $\Gamma_c(X;\mathscr{C}^\infty_X)$ in $\Gamma_c(X;\mathscr{D}b_X)$, hence defines $\mathscr{C}^\infty_X$ as a subsheaf of $\mathscr{D}b_X$.

Therefore, the sheaves $\mathscr{C}^{\infty,(j)}_X$ are naturally embedded into the sheaves $\mathscr{D}b^{(j)}_X$ of differential forms with distributions as coefficients and the differential on $\mathscr{D}b^{(j)}_X$ induces the differential on $\mathscr{C}^{\infty,(j)}_X$.

Notation 11.6.1. consider the complexes

$$\mathscr{C}^{\infty,(\bullet)}_X := 0 \to \mathscr{C}^{\infty,(0)}_X \xrightarrow{d} \cdots \to \mathscr{C}^{\infty,(n)}_X \to 0, \tag{11.6.1}$$

$$\mathscr{D}b^{(\bullet)}_X := 0 \to \mathscr{D}b^{(0)}_X \xrightarrow{d} \cdots \to \mathscr{D}b^{(n)}_X \to 0. \tag{11.6.2}$$

We call them the De Rham complexes on X with $\mathscr{C}^\infty$ and distributions coefficients, respectively.

Lemma 11.6.2. (The Poincaré lemma.) *Let $I = (]0,1[)^n$ be the unit open cube in $\mathbb{R}^n$. The complexes below are exact.*

$$0 \to \mathbb{C} \to \mathscr{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathscr{C}^{\infty,(n)}(I) \to 0,$$
$$0 \to \mathbb{C} \to \mathscr{D}b^{(0)}(I) \xrightarrow{d} \cdots \to \mathscr{D}b^{(n)}(I) \to 0.$$

Proof. We shall only treat the case of $\mathscr{C}^\infty(I)$. Consider the Koszul complex $K^\bullet(M,\varphi)$ over the ring $\mathbb{C}$, where $M = \mathscr{C}^\infty(I)$ and $\varphi = (\partial_1, \dots, \partial_n)$ (with $\partial_j = \frac{\partial}{\partial x_j}$). This complex is nothing but the complex:

$$0 \to \mathscr{C}^{\infty,(0)}(I) \xrightarrow{d} \cdots \to \mathscr{C}^{\infty,(n)}(I) \to 0.$$

Clearly $H^0(K^\bullet(M,\varphi)) \simeq \mathbb{C}$, and it is enough to prove that the sequence $(\partial_1,\dots,\partial_n)$ is coregular. Let $M_{j+1} = \ker(\partial_1)\cap\cdots\cap\ker(\partial_j)$. This is the space of $\mathscr{C}^\infty$-functions on I constant with respect to the variables $x_1,\dots,x_j$. Clearly, ∂_{j+1} is surjective on this space. ☐

The Poincaré lemma may be formulated intrinsically as:

Lemma 11.6.3. (The de Rham complex.) *Let X be a $\mathscr{C}^\infty$-manifold of dimension n. Then the natural morphisms $\mathbb{C}_X \to \mathscr{C}_X^{\infty,(\bullet)}$ and $\mathbb{C}_X \to \mathscr{D}b_X^{(\bullet)}$ are quasi-isomorphisms.*

We shall prove a finiteness and duality theorem for the cohomology of a compact manifold when the base ring $\mathbf{k}$ is the field $\mathbb{C}$. The duality result gives in this case an alternative proof of Corollary 11.5.6.

Theorem 11.6.4. (Poincaré duality on smooth manifolds.) *Assume X is compact. Then the $\mathbb{C}$-vector spaces $H^j(X;\mathbb{C}_X)$ and $H^{n-j}(X;or_X^{\mathbb{C}})$ are finite dimensional and dual one to each other.*

Proof. We shall make use of some results of functional analysis (refer to [Köt69]).

The vector spaces $\Gamma(X;\mathscr{C}_X^{\infty,(p)})$ are naturally endowed with a structure of Fréchet-Schwartz spaces (spaces of type FS), and the spaces $\Gamma(X;\mathscr{D}b_X^{(p)})$ are naturally endowed with a structure of dual of Fréchet-Schwartz spaces (spaces of type DFS). Set

$$\begin{aligned} E^\bullet &:= \Gamma(X;\mathscr{C}_X^{\infty,(\bullet)}),\\ F^\bullet &:= \Gamma(X;\mathscr{D}b_X^{(\bullet)}),\\ G^\bullet &:= \Gamma(X;\mathscr{D}b_X^{(\bullet)}\otimes \mathrm{or}_X). \end{aligned}$$

(i) Finiteness. The embedding $\mathscr{C}_X^{\infty,(j)} \hookrightarrow \mathscr{D}b_X^{(j)}$ defines the morphism of complexes $E^\bullet \to F^\bullet$. This morphism is continuous for the topologies of spaces FS and DFS and induces an isomorphism on the cohomology. This implies the finiteness of the vector spaces $H^j(E^\bullet)$.

(ii) Duality. Since the sheaf $\mathrm{or}_X^{\mathbb{C}}$ is locally isomorphic to $\mathbb{C}_X$, one gets the isomorphism

$$\mathrm{R}\Gamma(X;\mathrm{or}_X^{\mathbb{C}}) \xrightarrow{\sim} \Gamma(X;\mathscr{D}b_X^{(\bullet)}\otimes\mathrm{or}_X). \tag{11.6.3}$$

The topological vector spaces $\Gamma(X;C_X^{\infty,(p)})$ and $\Gamma(X;\mathscr{D}b_X^{(n-p)}\otimes\mathrm{or}_X)$ are naturally dual to each other, the pairing being defined by

$$(\varphi,u)\mapsto \int_X \varphi\cdot u.$$

This pairing is compatible to the differential:

$$(\varphi,du) = (d\varphi,u)$$

In other words, the two complexes $E^\bullet$ and $G^\bullet$ endowed with their topologies of vector spaces of type FS and DFS respectively are dual to each other. Since they have finite dimensional cohomology objects, this implies that the spaces $H^j(E^\bullet)$ and $H^{n-j}(G^\bullet)$ are dual to each other. ☐

Corollary 11.6.5. *Let X be a real compact connected manifold of dimension n. Then $H^n(X;\mathbb{C}_X)$ has dimension 0 or 1, and X is orientable if and only if this dimension is one.*

Proof. One has $H^0(X;\mathrm{or}_X) \neq 0$ if and only if or_X has a non identically zero global section, and if such a section exists, it will define a global isomorphism of or_X with $\mathbb{C}_X$. By the duality theorem, $H^0(X;\mathrm{or}_X)$ is the dual space to $H^n(X;\mathbb{C}_X)$. □

Cohomology of complex manifolds

Assume now that X is a complex manifold of complex dimension n, and let $X^{\mathbb{R}}$ be the real underlying manifold. The real differential d splits as $\partial + \bar{\partial}$, and one denotes by $\mathscr{C}_X^{\infty,(p,q)}$ the sheaf of $\mathscr{C}^\infty$ forms of type (p,q) with respect to $\partial, \bar{\partial}$. Consider the complexes

$$\mathscr{C}_X^{\infty,(p,\bullet)} := 0 \to \mathscr{C}_X^{\infty(p,0)} \xrightarrow{\bar{\partial}} \cdots \to \mathscr{C}_X^{\infty,(p,n)},$$
$$\mathscr{D}b_X^{(p,\bullet)} := 0 \to \mathscr{D}b^{(p,0)} \xrightarrow{\bar{\partial}} \cdots \to \mathscr{D}b_X^{(p,n)}.$$

Denote by Ω_X^p the sheaf of holomorphic p-forms. One usually sets

$$\Omega_X = \Omega_X^n.$$

The Dolbeault-Grothendieck lemma is formulated as:

Lemma 11.6.6. *Let X be a complex manifold. Then the natural morphisms $\Omega_X^p \to \mathscr{C}_X^{\infty,(p,\bullet)}$ and $\Omega_X^p \to \mathscr{D}b_X^{(p,\bullet)}$ are quasi-isomorphisms.*

Since the sheaves $\mathscr{C}_X^{\infty,(p,q)}$ and $\mathscr{D}b_X^{(p,q)}$ are c-soft, it follows that

$$\mathrm{R}\Gamma(X;\Omega_X^p) \xrightarrow{\sim} \Gamma(X;\mathscr{C}_X^{\infty,(p,\bullet)}) \xrightarrow{\sim} \Gamma(X;\mathscr{D}b_X^{(p,\bullet)}). \tag{11.6.4}$$
(11.6.5)

Theorem 11.6.7. (The Cartan-Serre finiteness and duality theorems.) *Let X be a compact manifold of complex dimension n. Then the $\mathbb{C}$-vector spaces $H^j(X;\Omega_X^p)$ and $H^{n-j}(X;\Omega_X^{n-p})$ are finite dimensional and dual one to each other.*

The proof goes as in the real case, recalling that a complex manifold is naturally oriented.

The Leray-Grothendieck integration morphism

Let $f\colon X \to Y$ be a morphism of complex manifolds. Denote by d_X (resp. d_Y) the complex dimension of X (resp. Y), and set for $l = d_X - d_Y \in \mathbb{Z}$. For $p,q \in \mathbb{Z}$ we have a natural morphism (inverse image of differential forms):

$$f^{-1}\mathscr{C}_Y^{\infty,(p,q)} \to \mathscr{C}_X^{\infty,(p,q)}$$

which commutes with $\bar{\partial}$ and defines by duality (recall that the complex manifolds X and Y are naturally oriented):

$$\int_f \colon f_!\mathscr{D}b_X^{(p,q)} \to \mathscr{D}b_Y^{(p-l,q-l)}. \tag{11.6.6}$$

These morphisms commute to $\bar\partial$ and define a morphism of complexes:

$$\begin{tikzcd}
\cdots \arrow[r] & f_!\mathscr{D}b_X^{(p,q)} \arrow[r, "\bar\partial"] \arrow[d, "\int_f"] & \mathscr{D}b_X^{(p,q+1)} \arrow[r] \arrow[d, "\int_f"] & \cdots \\
\cdots \arrow[r] & \mathscr{D}b_Y^{(p-l,q-l)} \arrow[r, "\bar\partial"] & \mathscr{D}b_Y^{(p-l,q-l+1)} \arrow[r] & \cdots
\end{tikzcd}$$

If one decides that $f_*\mathscr{D}b_X^{(p,d_X)}$ is in degree zero (hence, $\mathscr{D}b_Y^{(p-l,d_Y)}$ will also be in degree zero), the first line is quasi-isomorphic to $\mathrm{R}f_!\Omega_X^p\,[d_X]$ and the second line to $\Omega_Y^{p-l}\,[d_Y]$. Therefore we have constructed a morphism in $\mathrm{D^b}(\mathbb{C}_Y)$:

$$\mathrm{R}f_!\Omega_X^{p+d_X}\,[d_X] \to \Omega_Y^{p+d_Y}\,[d_Y].$$

In particular

Theorem 11.6.8. The residue morphism. *To each morphism $f : X \to Y$ of complex manifolds, the construction above defines functorially a morphism:*

$$\int_f : \mathrm{R}f_!\Omega_X\,[d_X] \to \Omega_Y\,[d_Y].$$

By "functorially", we mean that $\int_{\mathrm{id}_X} = \mathrm{id}$ and $\int_g \circ \int_f = \int_{g\circ f}$. In the absolute case we have thus obtained a map:

$$\int_X : H_c^{d_X}(X;\Omega_X) \to \mathbb{C}. \tag{11.6.7}$$

A cohomology class $u \in H_c^{d_X}(X;\Omega_X)$ may be represented by a distribution $v \in \Gamma_c(X;\mathscr{D}b_X^{(d_X,d_X)})$ modulo $\bar\partial w$ with $w \in \Gamma_c(X;\mathscr{D}b_X^{(d_X,d_X-1)})$. Since $\bar\partial w = (\bar\partial + \partial)w$, we get that $\int_X \bar\partial w = 0$ and $\int_X u$ is well defined. This is the required morphism.

If $X = \mathbb{C}$, we get in particular an integration map: $H^1_{\{0\}}(\mathbb{C};\Omega_{\mathbb{C}}) \to H^1_c(\mathbb{C};\Omega_{\mathbb{C}}) \to \mathbb{C}$, and one checks easily that, representing $H^1_{\{0\}}(\mathbb{C};\Omega_{\mathbb{C}})$ by $\Gamma(D\setminus\{0\};\Omega_{\mathbb{C}})/\Gamma(D;\Omega_{\mathbb{C}})$, where D is a disc centered at 0, the integral coincides, up to a non-zero factor, with the residue morphism.

Exercises to Chapter 11

Exercise 11.1. Let U be a convex open subset of $\mathbb{R}^d$. Prove that $\mathrm{R}\Gamma_c(U;\mathbf{k}_U)$ is concentrated in degree d and $H^d(U;\mathbf{k}_U) \simeq \mathbf{k}$.

Exercise 11.2. Let X be a locally compact space. Prove the isomorphisms $H_c^j(X;F) \simeq \operatorname*{colim}_K H_K^j(X;F)$, where K ranges over the family of compact subsets of X.

Exercise 11.3. (i) Let $I = [0,1[$. Show that $\mathrm{R}\Gamma_c(I;\mathbf{k}_I) = 0$.

(ii) Let s denote the map $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x+y$. Let $D \subset \mathbb{R}^2; D =]-1,1[\times[-1,1]$. Calculate $\mathrm{R}s_!(\mathbf{k}_D)$.

Exercise 11.4. Let $Y = [0,1]\times]0,1[$ and let X denote the manifold obtained by identifying $(0,t)$ and $(1,1-t)$. Let S denote the hypersurface of X, the image of the diagonal of Y. Calculate $\Gamma(X;\mathrm{or}_{S/X})$.

Exercise 11.5. Let X be a locally compact space and let $\{F_i\}_{i\in I}$ be an inductive system of c-soft sheaves on X, with I filtered. Prove that $\operatorname*{colim}_i F_i$ is c-soft.

Exercise 11.6. (i) Let t be an indeterminate, and denote by $F[t]$ the sheaf $F \otimes (\mathbf{k}[t])_X$. Prove that on $X = \mathbb{C}$, the sequence of sheaves $0 \to \mathscr{O}_X[t] \to \mathscr{C}^\infty_X[t] \xrightarrow{\bar\partial} \mathscr{C}^\infty_X[t] \to 0$ is exact.

(ii) Using the fact that there are $\mathscr{C}^\infty$-functions φ with compact support such that the support of any solution of the equation $\bar\partial\psi = \varphi$ is the whole set X, deduce that $H^1(\mathbb{C};\mathscr{O}_{\mathbb{C}}[t]) \neq 0$.

Exercise 11.7. Recall that $f : Z \to X$ is a trivial covering if there exists a non empty set S, a topological isomorphism $h : Z \xrightarrow{\sim} X \times S$ where S is endowed with the discrete topology, such that $f = p \circ h$ where $p : X \times S \to X$ is the projection. Also recall that $f : Z \to X$ is a locally trivial covering if f is surjective and any $x \in X$ has an open neighborhood U such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a trivial covering.

Prove that is $f\colon Z \to X$ is a locally trivial covering, then the functor f^{-1} is right adjoint to $f_!$.

Exercise 11.8. Assume $f\colon X \to Y$ is a covering. Prove that $f^! \simeq f^{-1}$. (Hint: the functor $f_!$ is a left adjoint to the exact functor f^{-1} and we get the isomorphism $\operatorname{Hom}(F, f^{-1}G) \simeq \operatorname{Hom}(F, f^!G)$ for all c-soft sheaf F.)

Exercise 11.9. Let $\mathbb{S}^n$ denote the real n-dimensional sphere, $\mathbb{P}^n$ the real n-dimensional projective space, $\gamma : \mathbb{S}^n \to \mathbb{P}^n$ the natural projection. Prove that γ is a 2-covering and deduce that for $n \geq 2$ there are at least two different locally constant sheaves of rank one on $\mathbb{P}^n$.

Exercise 11.10. Let $X = \mathbb{R}$ and $\mathbf{k} = \mathbb{Q}$. Consider the sets $A = \{1/n; n \in \mathbb{N}_{>0}\}$ and $Z = A \cup \{0\}$. Prove that the sheaf $\mathbf{k}_A$ is soft and that $H^2_{\{0\}}(X;\mathbf{k}_{X\setminus Z}) \neq 0$. (See [KS90, Exe. III.1].)

Chapter 12

Constructible sheaves and the subanalytic topology

In this last chapter, we briefly study constructible sheaves on a real analytic manifold. We first introduce the class of subanalytic subsets and the notion of being subanalytic up to infinity.

12.1 Subanalytic subsets

12.2 Subanalytic geometry

12.3 Constructible sheaves

12.4 Subanalytic topology

Bibliography

[AM69] Michael Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Weisley, 1969.

[AW67] Michael Atiyah and Charles T.C. Wall, *Cohomology of groups*, Algebraic Number Theory, J. W. S. Cassels and A. Frölich eds, Academic Press, 1967, pp. 94–114.

[BG72] Marcel Berger and Bernard Gostiaux, *Géométrie différentielle*, Armand Colin, 1972.

[Bor94] Francis Borceux, *Handbook of categorical algebra I, II, III*, Encyclopedia of Mathematics and itsApplications, vol. 51, Cambridge University Press, 1994.

[BT82] Raoul Bott and Loring W Tu, *Differential forms in algebraic topology*, Graduate Texts in Math., vol. 82, Springer, 1982.

[Bou80] Nicolas Bourbaki, *Elements de Mathématiques, Algèbre Ch 10*, Masson, 1980.

[Bre67] Glen E. Bredon, *Sheaf Theory*, McGraw-Hill, 1967.

[Bro82] Kenneth S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer, 1982.

[CE56] Henri Cartan and Samuel Eilenberg, *Homological algebra*, Princeton University Press, 1956.

[Che55] Claude Chevalley, *The construction and study of certain important algebras*, Publ. Soc. Math. Japan, 1955.

[Cis19] Denis-Charles Cisinsky, *Higher categories and homotopical algebra*, Cambridge Studies in Advanced Mathematics, vol. 180, Cambridge University Press, 2019.

[Del66] Pierre Deligne, *Cohomologie à support propre et construction du foncteur* $f^!$, Residues and Duality, Springer, Berlin, 1966, pp. 404–421. Lecture Notes in Math., Vol. 20.

[dR60] Georges de Rham, *Variétés différentiables*, Hermann, 1960.

[Dim04] Alexandru Dimca, *Sheaves in topology*, Universitext, Springer, 2004.

[EGA3] Alexander Grothendieck, *Elements de géométrie algébrique III*, Vol. 11, Publ. IHES, 1961.

[EML45] Samuel Eilenberg and Saunders Mac Lane, *General theory of natural equivalences*, Trans. Amer. Math. Soc. 58 (1945), 231–294.

[Fre64] Peter Freyd, *Abelian categories*, Harper & Row, 1964.

[GZ67] Peter Gabriel and Michel Zisman, *Calcul of fractions and homotopy theory*, Springer, 1967.

[GM96] Serguei Gelfand and Yuri Manin, *Methods of homological algebra*, Springer, 1996.

[God58] Roger Godement, *Topologie algébrique et théorie des faisceaux*, Hermann, 1958.

[Gro57] Alexander Grothendieck, *Sur quelques points d'algèbre homologique,* Tohoku Math. Journ. (1957), 119–183, available at `translated~by~M.~L~Barr~and~M.~Barr~https://cupdf.com/document/grothendieck-tohoku-eng.html?page=1`.

[Gui19] Stéphane Guillermou, *Sheaves and symplectic geometry of cotangent bundles* (2019), available at `arXiv:1905.07341`.

[GS16] Stéphane Guillermou and Pierre Schapira, *Construction of sheaves on the subanalytic site*, in: Subanalytic sheaves and Sobolev spaces, Astérisque, Soc. Math. France, 2016, pp. 4–60.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture Notes in Math., vol. 20, Springer, 1966.

[Hör66] Lars Hörmander, *An introduction to complex analysis*, Van Norstrand, 1966.

[Hör83] Lars Hörmander, *The analysis of linear partial differential operators I,II*, Grundlehren der Math. Wiss, vol. 256, 257, Springer-Verlag, 1983.

[Ive86] Birger Iversen, *Cohomology of sheaves*, Universitex, Springer, 1986.

[Kas03] Masaki Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003.

[Krö07] Ralf Krömer, *Tool and object*, Science Networks. Historical Studies, vol. 32, Birkhäuser Verlag, Basel, 2007.

[KS96] Masaki Kashiwara and Pierre Schapira, *Moderate and formal cohomology associated with constructible sheaves*, Mémoires Soc. Math. France, vol. 64, 1996.

[KS82] ______, *Micro-support des faisceaux: applications aux modules différentiels*, C. R. Acad. Sci. Paris **295, 8** (1982), 487–490.

[KS85] ______, *Microlocal study of sheaves*, Astérisque, vol. 128, Soc. Math. France, 1985.

[KS90] ______, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, vol. 292, Springer, 1990.

[KS01] ______, *Ind-Sheaves*, Astérisque, vol. 271, Soc. Math. France, 2001.

[KS06] ______, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften, vol. 332, Springer-Verlag, Berlin, 2006.

[Kel06] Bernhard Keller, *On differential graded categories*, ICM, vol. II, Eur. Math. Soc. Zúrich, 2006.

[Köt69] G Köthe, *Topological vector spaces*, Grundlehren der Math. Wiss., Springer-Verlag, Berlin, 1969.

[Lan21] Markus Land, *Introduction to Infinity-Categories*, Birkhauser, Springer Nature, 2021.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Lur10] ______ (2010), available at `www.math.ias.edu/~lurie/`.

[Mac98] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Math., Springer, 1998.

[MM92] Saunders MacLane and Moerdijk, *Sheaves in geometry and logic*, Universitext., Springer, 1992.

[May99] J. Peter May, *A concise course in algebraic topology*, Chicago Lectures in Mathematics, The University of Chicago Press, 1999.

[Mit60] B Mitchell, *dissertation, Brown University* (1960).

[NZ09] David Nadler and Eric Zaslow, *Constructible sheaves and the Fukaya category*, J. Amer. Math. Soc. **22** (2009), 233–286.

[Nee01] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, Princeton University Press, 2001.

[RV22] Emily Riehl and Dominic Verity, *Elements of ∞-Category Theory*, Cambridge University Press, 2022.

[Sat59] Mikio Sato, *Theory of hyperfunctions, I & II*, Journ. Fac. Sci. Univ. Tokyo **8** (1959, 1960), 139–193, 487–436.

[SKK73] Mikio Sato, Takahiro Kawai, and Masaki Kashiwara, *Microfunctions and pseudo-differential equations*, Hyperfunctions and pseudo-differential equations (Proc. Conf., Katata, 1971; dedicated to the memory of André Martineau), Springer, Berlin, 1973, pp. 265–529. Lecture Notes in Math., Vol. 287.

[Sch21] Pierre Schapira, *Microlocal analysis and beyond*, New Spaces in Mathematics (Mathieu Anel and Gabriel Catren, eds.), Cambridge University Press, 2021.

[Schu03] Jörg Schürmann, *Topology of singular spaces and constructible sheaves*, Monografie Matematyczne, vol. 63, Springer Basel AG, 2003.

[SGA4] Mike Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Théorie des topos et cohomologie étale des schémas*, Sém. Géom. Alg. (1963–64), Lecture Notes in Math., vol. 269, 270, 305, Springer, Berlin, 1972/73.

[Tam08] Dmitry Tamarkin, *Microlocal conditions for non-displaceability* (2008), available at `arXiv:0809.1584`.

[Tam94] Günter Tamme, *Introduction to étale cohomology*, Universitex, Springer, 1994.

[Toë14] Bertrand Toën, *Derived algebraic geometry*, EMS Surv. Math, Europ. Math. Soc., 2014.

[Ver69] Jean-Louis Verdier, *Théorèmes de dualité pour les espaces localement compacts* (Publ. Inst. Rech. Math. Av., ed.), Heidelberg-Strasbourg seminar 1966–67 exp. 4, 1969.

[Ver96] ______, *Des catégories dérivées des catégories abéliennes*, Astérisque, vol. 239, Soc. Math. France, 1996.

[Wei94] Charles Weibel, *An introduction to homological algebra*, Vol. 38, Cambridge Studies in Advanced Math., 1994.

[Yek20] Amnon Yekutieli, *Derived categories*, Cambridge study in advanced mathematics, vol. 183, Cambridge University Press, 2020.